Multiport Communications

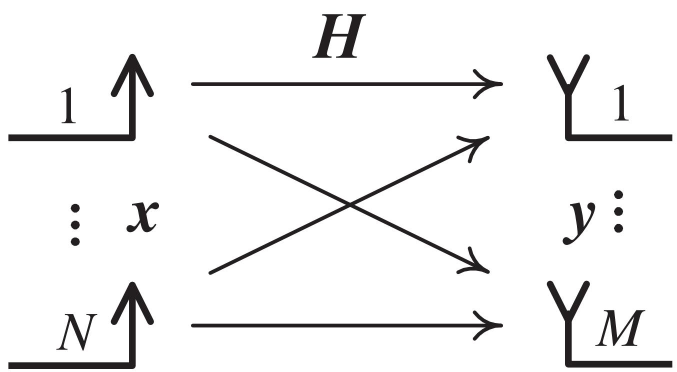
Towards a Circuit Theory of Communication

Josef A. Nossek

Napoli, 10 giugno 2010

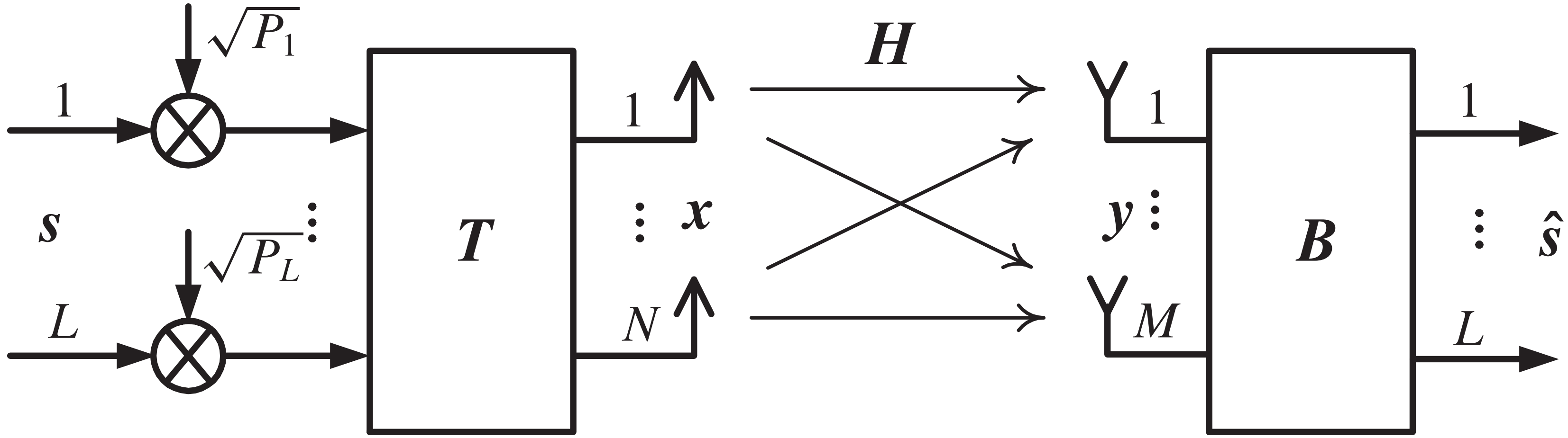
Technische Universität München
Institute for Circuit Theory and Signal Processing

Multi-channel radio communication



$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}$$

Multi-channel radio communication

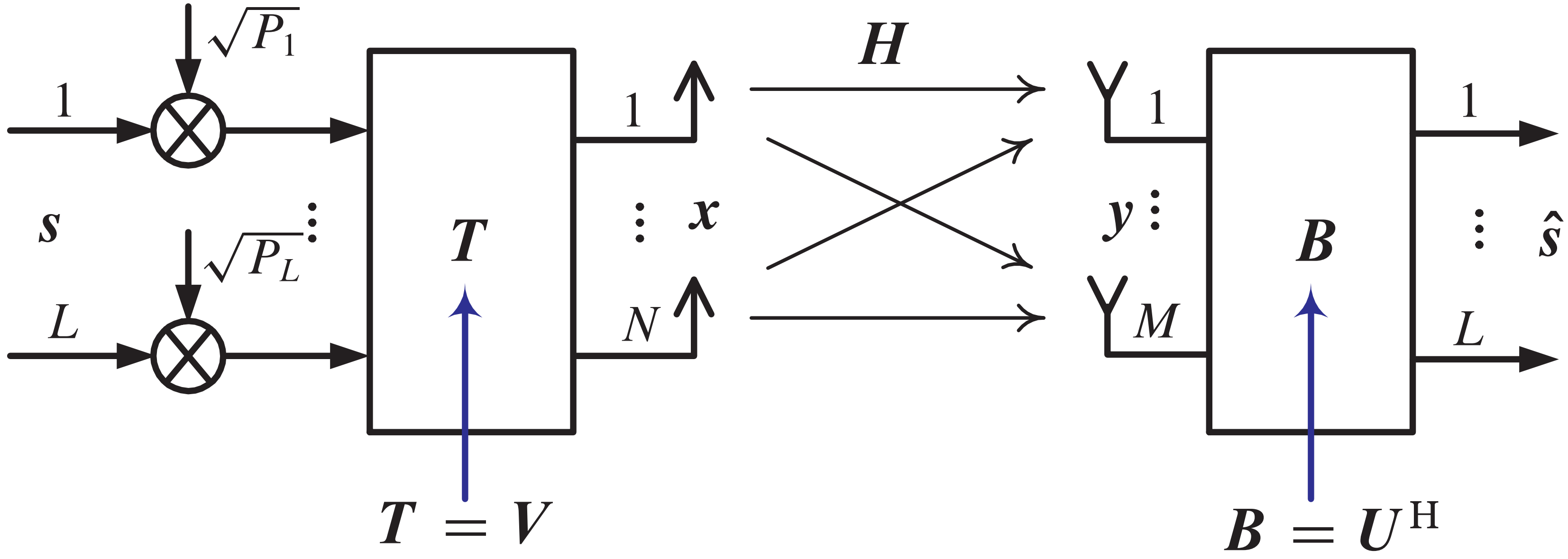


$$\hat{\boldsymbol{s}} = \boldsymbol{B}\boldsymbol{H}\boldsymbol{T}\boldsymbol{P}^{1/2}\boldsymbol{s} + \boldsymbol{B}\boldsymbol{n}$$

$$L = \operatorname{rank}(\boldsymbol{H}) \leq \min(M, N)$$

$$\boldsymbol{P} = \operatorname{diag}\{P_i\}_{i=1}^{L}$$

Multi-channel radio communication



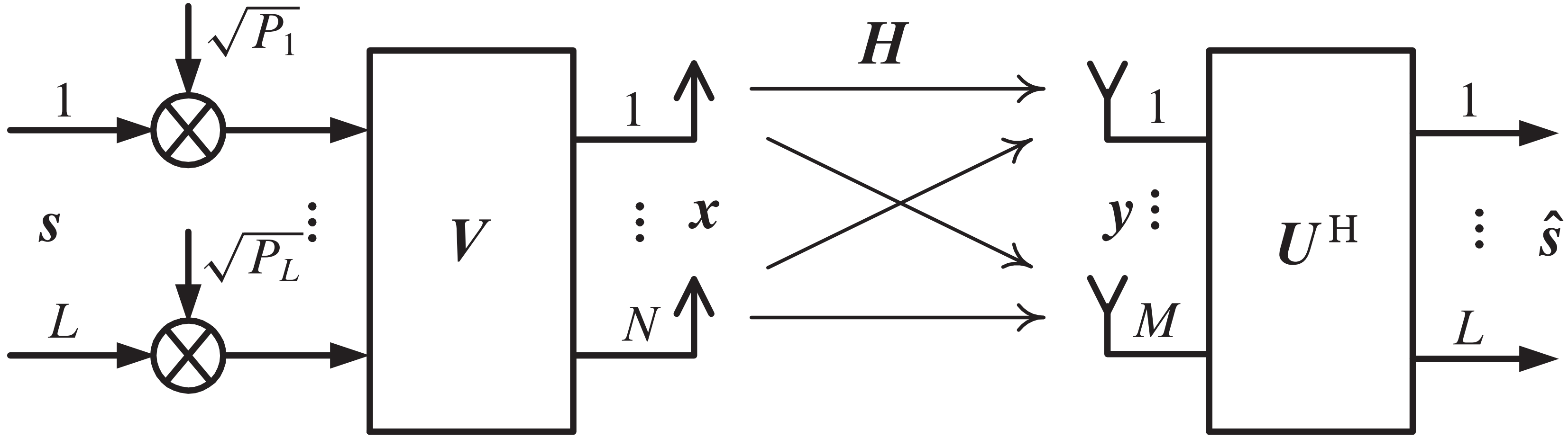
Singular Value Decomposition

$$\hat{\boldsymbol{s}} = \boldsymbol{B}\boldsymbol{H}\boldsymbol{T}\boldsymbol{P}^{1/2}\boldsymbol{s} + \boldsymbol{B}\boldsymbol{n}$$

$$\hat{\boldsymbol{s}} = \boldsymbol{B}\ \boldsymbol{U}\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_L \end{bmatrix}\boldsymbol{V}^{\mathrm{H}}\ \boldsymbol{T}\boldsymbol{P}^{1/2}\boldsymbol{s} + \boldsymbol{n}'$$

$$L = \operatorname{rank}(\boldsymbol{H}) \leq \min(M, N)$$

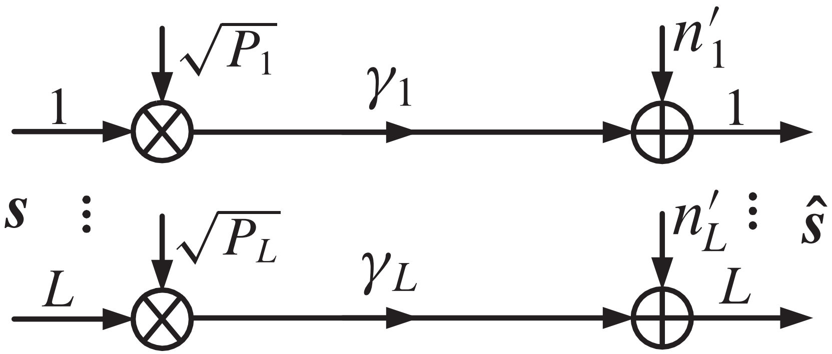
$$\boldsymbol{P} = \operatorname{diag}\{P_i\}_{i=1}^{L}$$



Singular Value Decomposition

$$\hat{\boldsymbol{s}} = \boldsymbol{B}\boldsymbol{H}\boldsymbol{T}\boldsymbol{P}^{1/2}\boldsymbol{s} + \boldsymbol{B}\boldsymbol{n}$$

$$= \mathbf{I}\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_L \end{bmatrix}\mathbf{I}\,\boldsymbol{P}^{1/2}\boldsymbol{s} + \boldsymbol{n}'$$



$$L = \operatorname{rank}(\boldsymbol{H})$$

$$\leq \min(M, N)$$

$$\boldsymbol{P} = \operatorname{diag}\{P_i\}_{i=1}^{L}$$

* Scalar signal: $x(t), \quad \mathbb{R} \mapsto \mathbb{C}$

- Scalar signal: $x(t), \quad \mathbb{R} \mapsto \mathbb{C}$

- Signal energy: $E = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t$

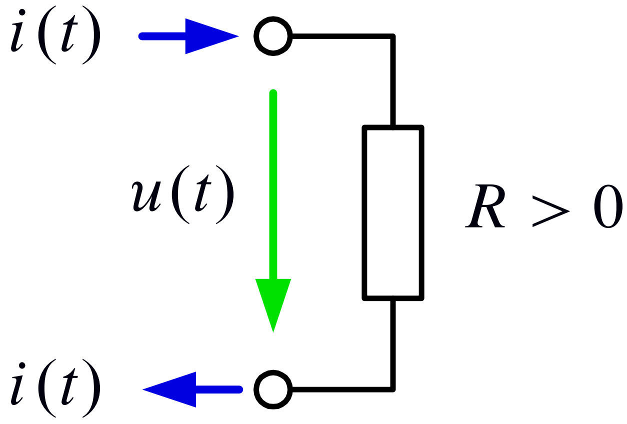
- Scalar signal: $x(t), \quad \mathbb{R} \mapsto \mathbb{C}$

- Signal energy: $E = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t$

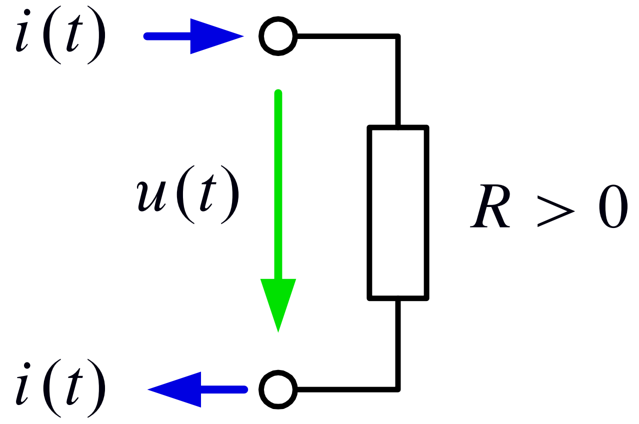
- Signal power: $P = |x(t)|^2$

Energy of scalar signals

- Scalar signal: $x(t), \quad \mathbb{R} \mapsto \mathbb{C}$
- Signal energy: $E = \int\limits_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t$
- Signal power: $P = |x(t)|^2$
- Passive, linear oneport:



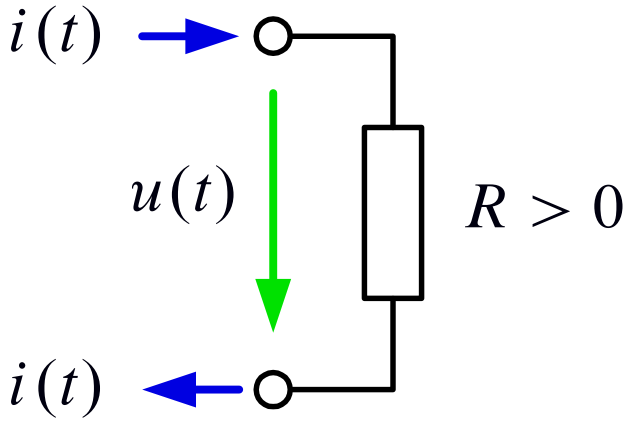
- Scalar signal: $x(t), \quad \mathbb{R} \mapsto \mathbb{C}$
- Signal energy: $E = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t$
- Signal power: $P = |x(t)|^2$
- Passive, linear oneport:



- Ohm's law: $u(t) = R\,i(t)$

Energy of scalar signals

- Scalar signal: $x(t), \quad \mathbb{R} \mapsto \mathbb{C}$
- Signal energy: $E = \int\limits_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t$
- Signal power: $P = |x(t)|^2$
- Passive, linear oneport:



- Ohm's law: $u(t) = R\,i(t)$
- Physical power:

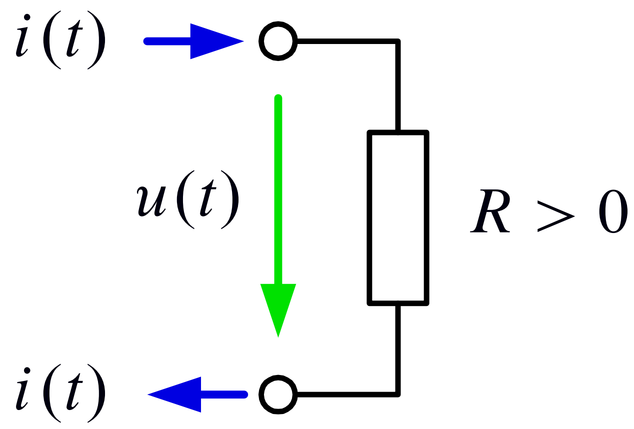
$$P_{\text{phy}} = u(t) \cdot i(t)$$
$$= u^2(t)/R = i^2(t) \cdot R$$

Energy of scalar signals

- Scalar signal: $x(t), \quad \mathbb{R} \mapsto \mathbb{C}$
- Signal energy: $E = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t$
- Signal power: $P = |x(t)|^2$

- Signal assignment:
  1. $x(t) \sim u(t) \implies P \sim P_{\text{phy}}$
  2.
  3.

- Passive, linear oneport:



- Ohm's law: $u(t) = R\,i(t)$
- Physical power:

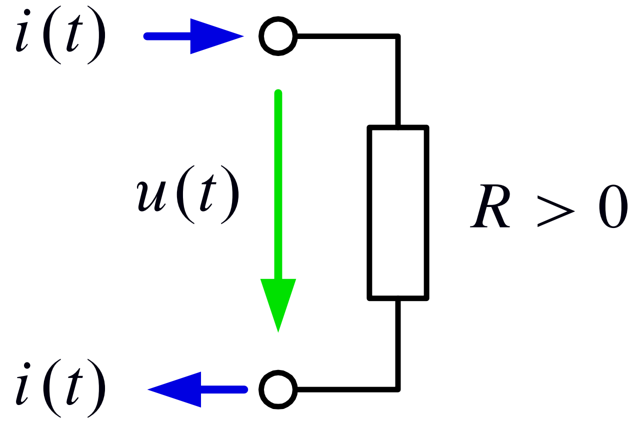
$$P_{\text{phy}} = u(t) \cdot i(t)$$
$$= u^2(t)/R = i^2(t) \cdot R$$

- Scalar signal: $x(t), \quad \mathbb{R} \mapsto \mathbb{C}$
- Signal energy: $E = \int\limits_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t$
- Signal power: $P = |x(t)|^2$

- Signal assignment:

1. $x(t) \sim u(t) \implies P \sim P_{\text{phy}}$
2. $x(t) \sim i(t) \implies P \sim P_{\text{phy}}$
3.

- Passive, linear oneport:



- Ohm's law: $u(t) = R\,i(t)$
- Physical power:

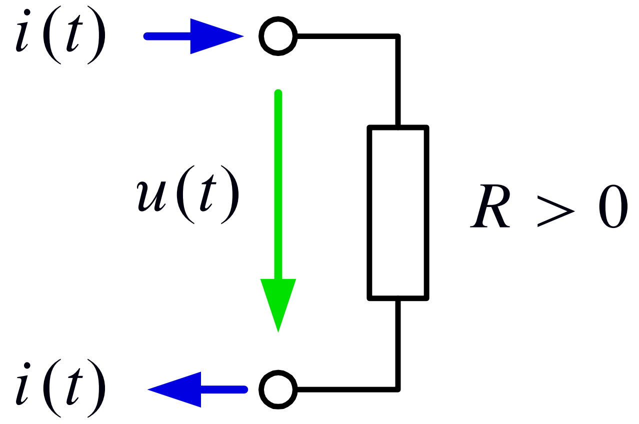
$$P_{\text{phy}} = u(t) \cdot i(t)$$
$$= u^2(t)/R = i^2(t) \cdot R$$

Energy of scalar signals

- Scalar signal: $x(t), \quad \mathbb{R} \mapsto \mathbb{C}$
- Signal energy: $E = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t$
- Signal power: $P = |x(t)|^2$

- Signal assignment:
  1. $x(t) \sim u(t) \implies P \sim P_{\text{phy}}$
  2. $x(t) \sim i(t) \implies P \sim P_{\text{phy}}$
  3. $x(t) \sim \big(\alpha u(t) + \beta i(t)\big) \implies P \sim P_{\text{phy}}$

- Passive, linear oneport:



- Ohm's law: $u(t) = R\, i(t)$
- Physical power:

$$P_{\text{phy}} = u(t) \cdot i(t) = u^2(t)/R = i^2(t) \cdot R$$

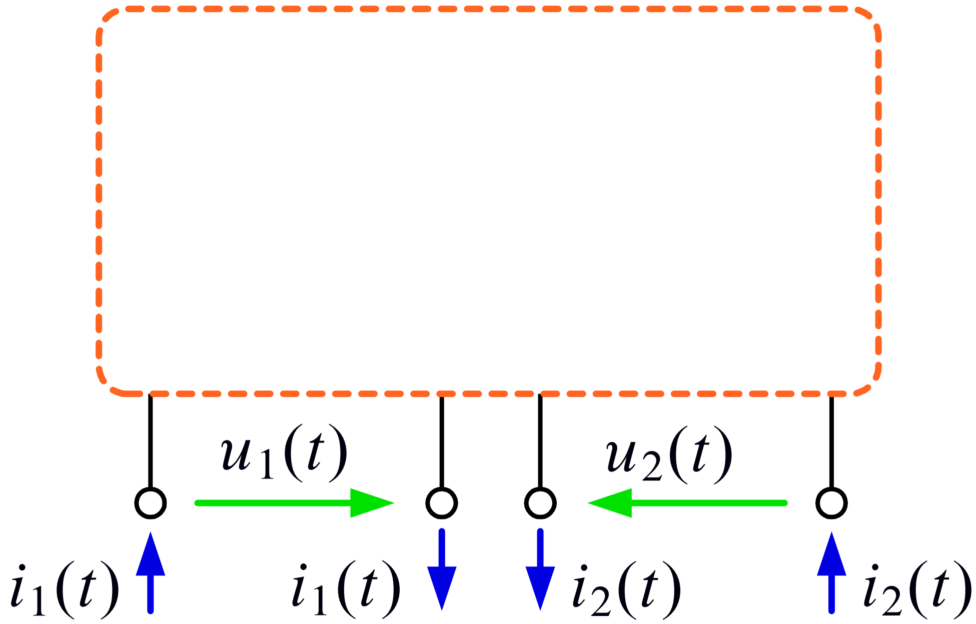
- Vector signal: $\boldsymbol{x}(t) = \left[x_1(t) \;\; x_2(t)\right]^{\mathrm{T}}$

Energy of vector signals

- Vector signal: $\boldsymbol{x}(t) = \left[x_1(t)\ \ x_2(t)\right]^{\mathrm{T}}$
- Signal energy: $E = \int\limits_{-\infty}^{\infty} ||\boldsymbol{x}(t)||_2^2 \,\mathrm{d}t$
- Signal power: $P = ||\boldsymbol{x}(t)||_2^2$

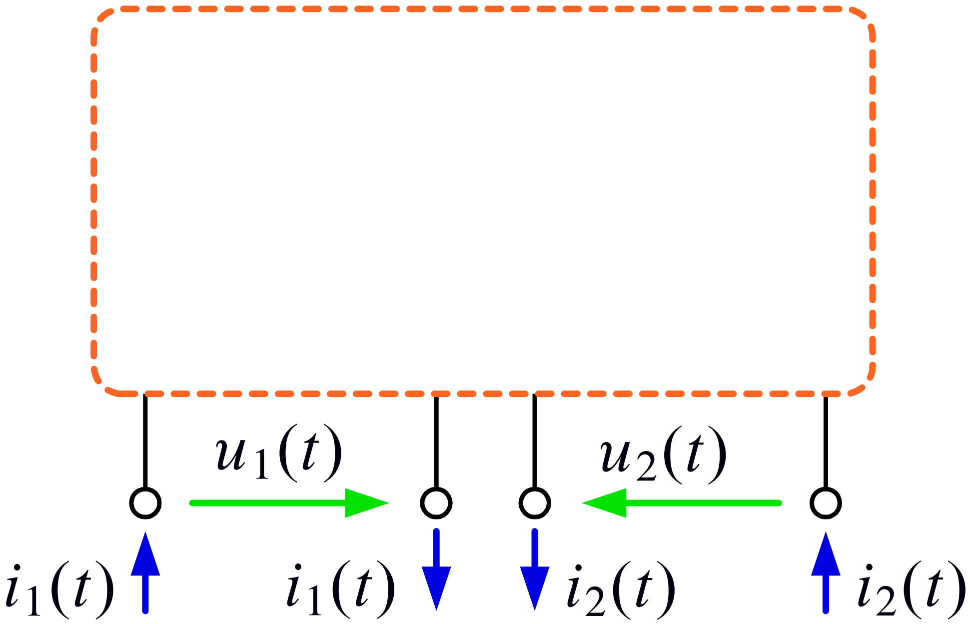
Energy of vector signals

- Vector signal: $\boldsymbol{x}(t) = \left[x_1(t)\ x_2(t)\right]^{\mathrm{T}}$
- Signal energy: $E = \int\limits_{-\infty}^{\infty} ||\boldsymbol{x}(t)||_2^2 \,\mathrm{d}t$
- Signal power: $P = ||\boldsymbol{x}(t)||_2^2$
- Passive, linear twoport:



Energy of vector signals

- Vector signal: $\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^{\mathrm{T}}$
- Signal energy: $E = \int_{-\infty}^{\infty} ||\boldsymbol{x}(t)||_2^2 \, \mathrm{d}t$
- Signal power: $P = ||\boldsymbol{x}(t)||_2^2$
- Passive, linear twoport:



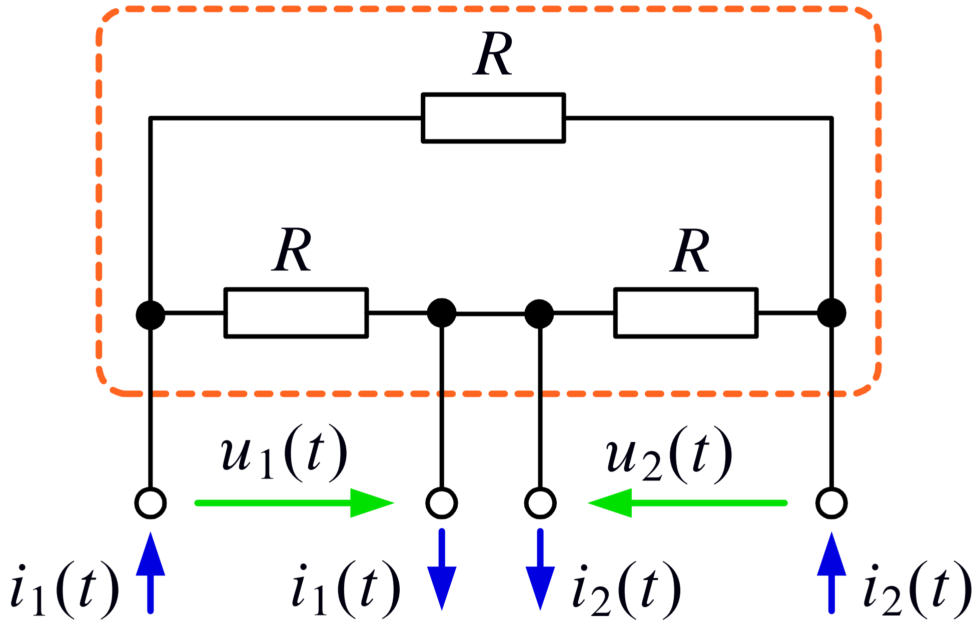
- Physical power:

$$P_{\text{phy}} = u_1(t)i_1(t) + u_2(t)i_2(t)$$

Energy of vector signals

- Vector signal: $\boldsymbol{x}(t) = \left[x_1(t)\;\; x_2(t)\right]^{\mathrm{T}}$
- Signal energy: $E = \int_{-\infty}^{\infty} ||\boldsymbol{x}(t)||_2^2 \, \mathrm{d}t$
- Signal power: $P = ||\boldsymbol{x}(t)||_2^2$

- Passive, linear twoport:



- Physical power:

$$P_{\text{phy}} = u_1(t)i_1(t) + u_2(t)i_2(t)$$

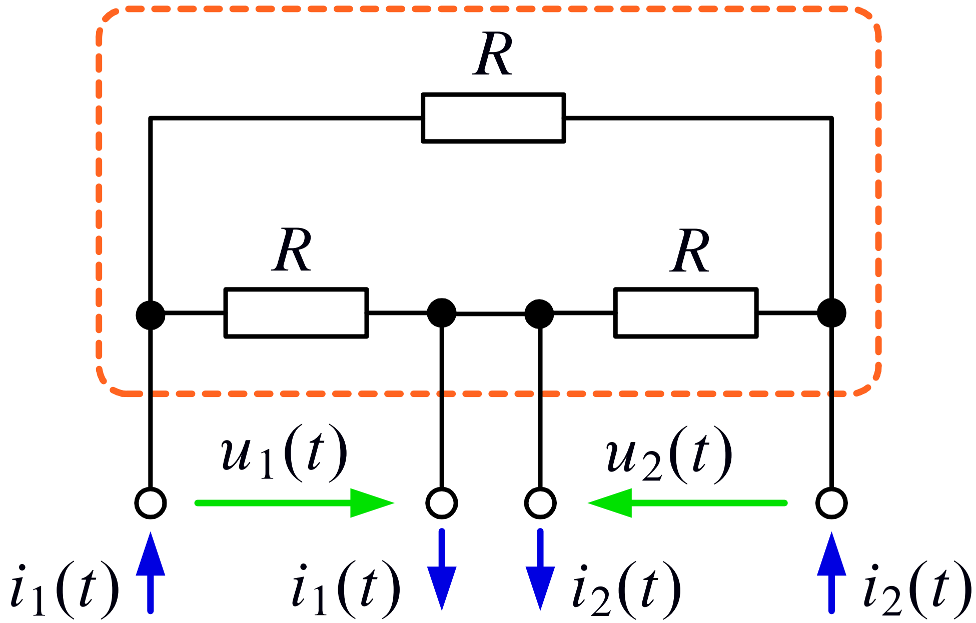
$$= \frac{u_1^2(t)+u_2^2(t)-u_1(t)u_2(t)}{R/2}$$

- Vector signal: $\boldsymbol{x}(t) = \left[x_1(t)\ \ x_2(t)\right]^{\mathrm{T}}$
- Signal energy: $E = \int_{-\infty}^{\infty} ||\boldsymbol{x}(t)||_2^2\,\mathrm{d}t$
- Signal power: $P = ||\boldsymbol{x}(t)||_2^2$

- Signal assignment:

1. $\boldsymbol{x}(t) \sim \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad \Longrightarrow \quad P \not\sim P_{\text{phy}}$

- Passive, linear twoport:



- Physical power:

$$P_{\text{phy}} = u_1(t)i_1(t) + u_2(t)i_2(t)$$

$$= \frac{u_1^2(t) + u_2^2(t) - u_1(t)u_2(t)}{R/2}$$

- Vector signal: $\boldsymbol{x}(t) = \left[x_1(t)\ \ x_2(t)\right]^{\mathrm{T}}$
- Signal energy: $E = \int_{-\infty}^{\infty} \|\boldsymbol{x}(t)\|_2^2 \, \mathrm{d}t$
- Signal power: $P = \|\boldsymbol{x}(t)\|_2^2$

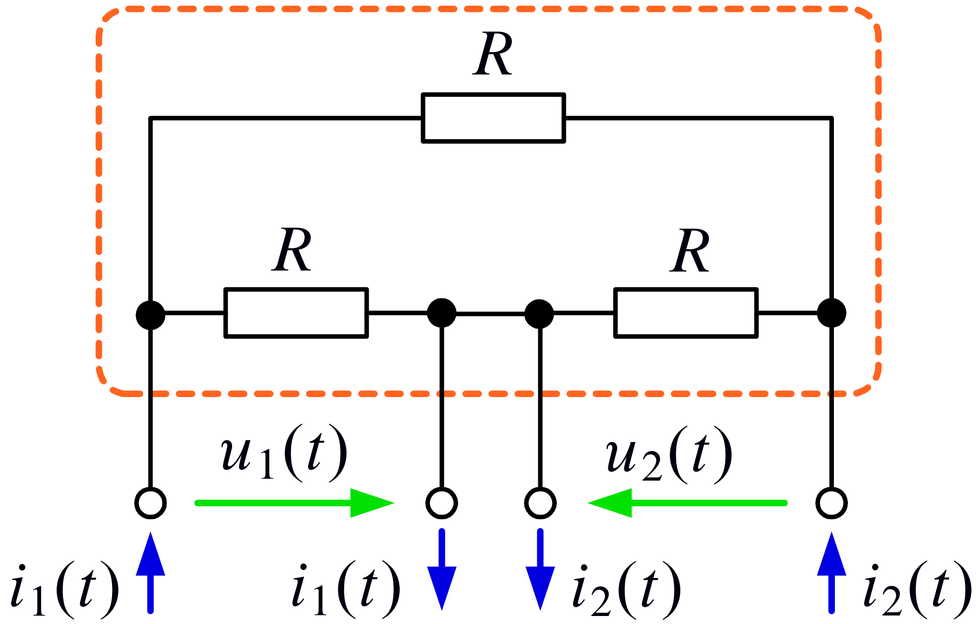
- Signal assignment:

1. $\boldsymbol{x}(t) \sim \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \implies P \not\sim P_{\text{phy}}$

mit $\alpha = \sqrt{3} - 2$, und

2. $\boldsymbol{x}(t) \sim \begin{bmatrix} u_1(t) + \alpha u_2(t) \\ \alpha u_1(t) + u_2(t) \end{bmatrix} \implies P \sim P_{\text{phy}}$

- Passive, linear twoport:



- Physical power:

$$P_{\text{phy}} = u_1(t)i_1(t) + u_2(t)i_2(t)$$

$$= \frac{u_1^2(t) + u_2^2(t) - u_1(t)u_2(t)}{R/2}$$

The question:

»What does the signal $x(t)$ mean physically?«

is in the scalar case only of secondary interest.

However, for *vector signals* it is *essential!*

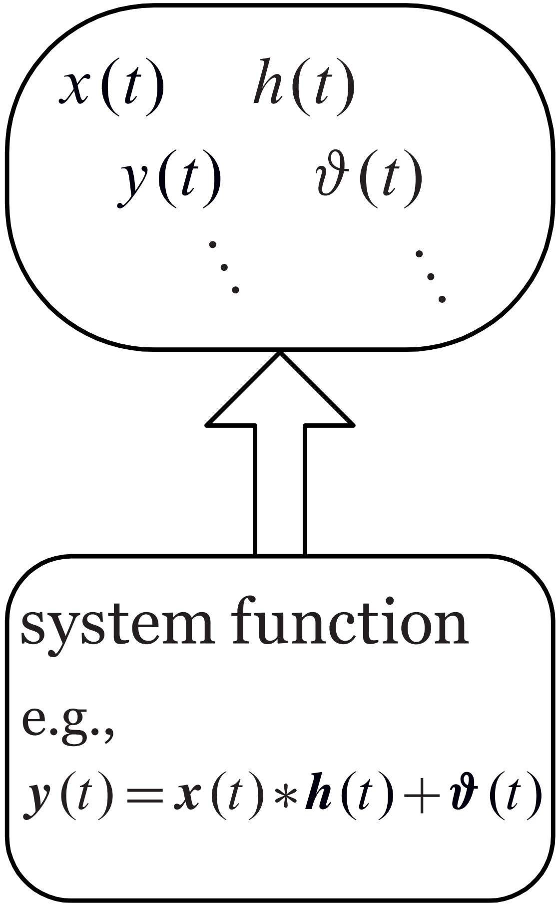
Signals

$x(t)$ $h(t)$

$y(t)$ $\vartheta(t)$

$\ddots$ $\ddots$

Signals


$x(t)$ $h(t)$
$y(t)$ $\vartheta(t)$
system function
e.g.,
$\boldsymbol{y}(t) = \boldsymbol{x}(t) * \boldsymbol{h}(t) + \boldsymbol{\vartheta}(t)$

- **Signals**

$x(t)$ $h(t)$

$y(t)$ $\vartheta(t)$

$\ddots$ $\ddots$

system function

e.g.,

$\boldsymbol{y}(t) = \boldsymbol{x}(t) * \boldsymbol{h}(t) + \boldsymbol{\vartheta}(t)$

- **Physical quantities**

$u(t)$ $\vec{E}(t)$

$i(t)$ $\vec{B}(t)$

$\ddots$ $\ddots$

Signal theory and physics

Signals

$x(t)$ $h(t)$

$y(t)$ $\vartheta(t)$

$\ddots$ $\ddots$

system function

e.g.,

$$\boldsymbol{y}(t) = \boldsymbol{x}(t) * \boldsymbol{h}(t) + \boldsymbol{\vartheta}(t)$$

Physical quantities

$u(t)$ $\vec{E}(t)$

$i(t)$ $\vec{B}(t)$

$\ddots$ $\ddots$

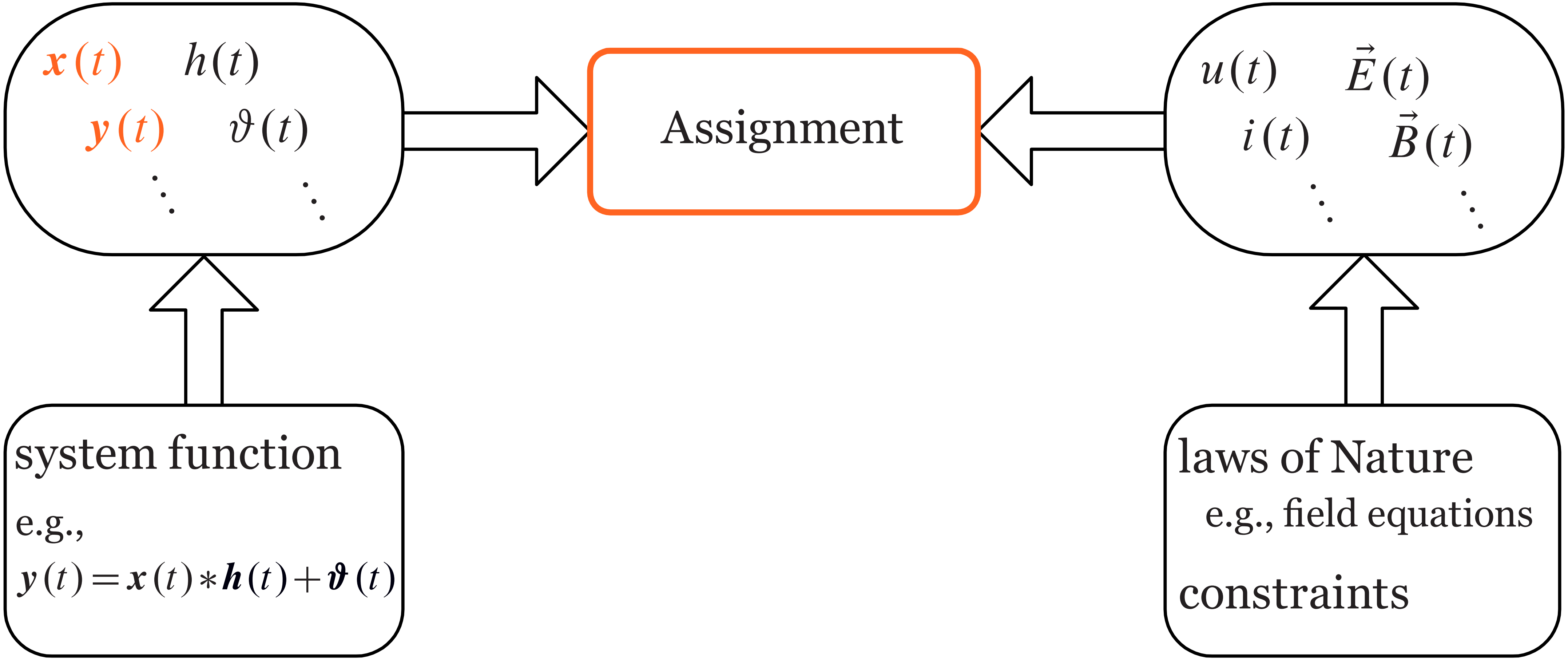
laws of Nature

e.g., field equations

constraints

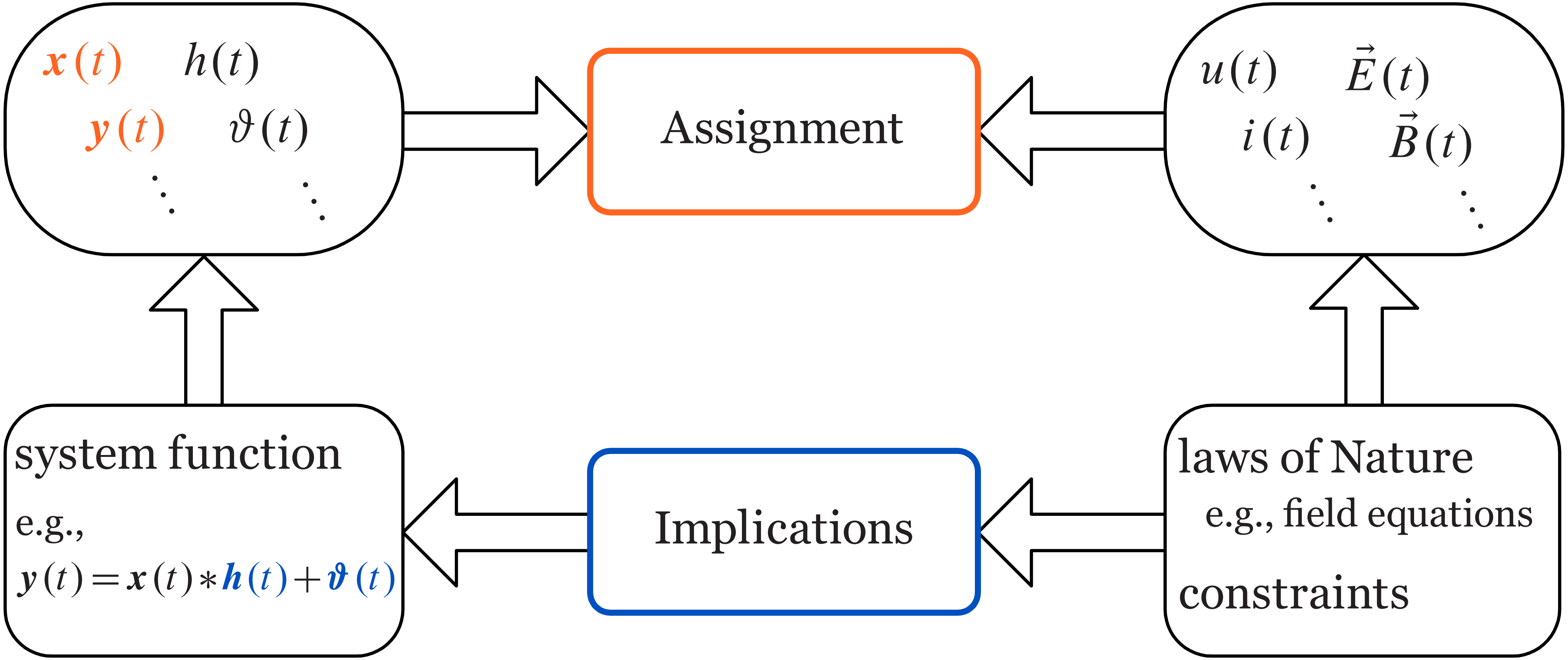
Signal theory and physics

- Signals
- Physical quantities



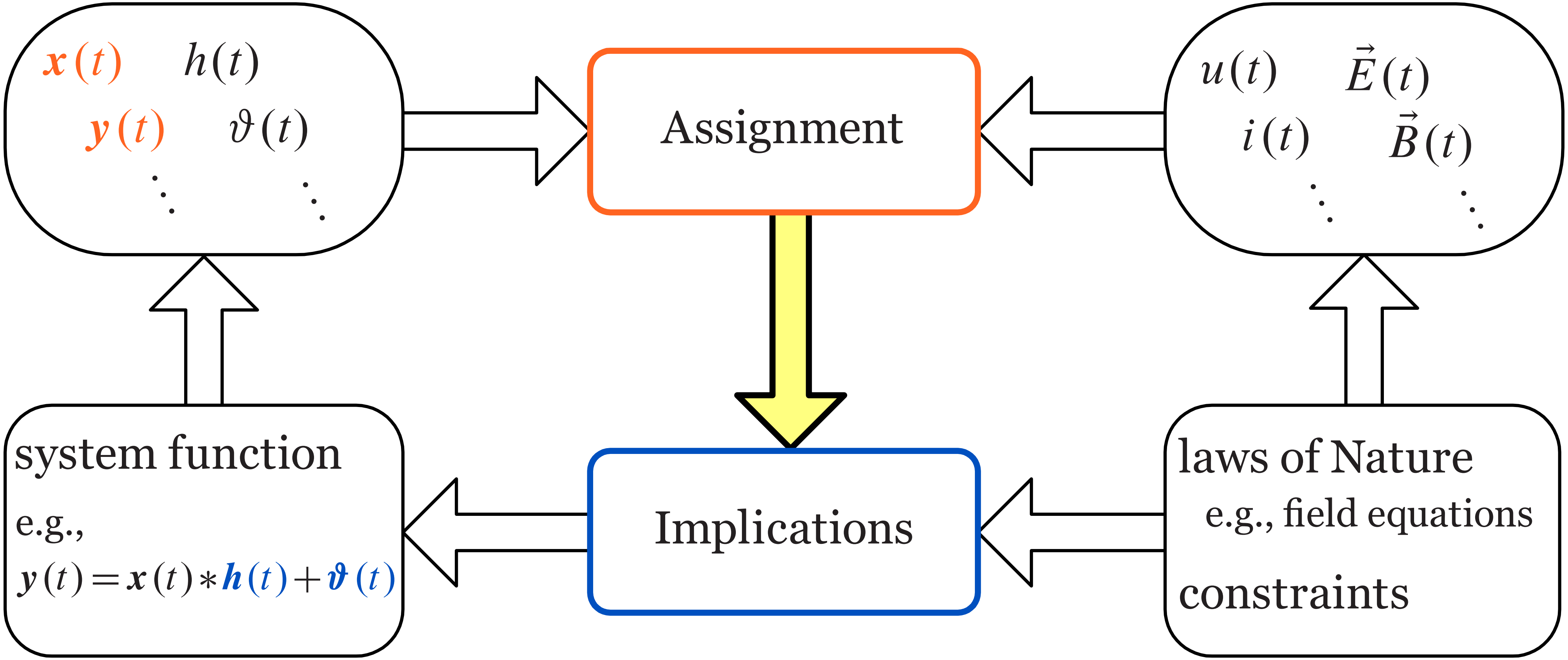
Signal theory and physics

- Signals
- Physical quantities



Signal theory and physics

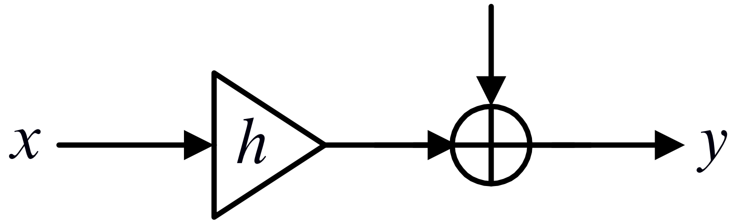
- Signals
- Physical quantities



All relevant implications for the system function have to be figured out!

Learning an implication by example

- AWGN channel: ϑ

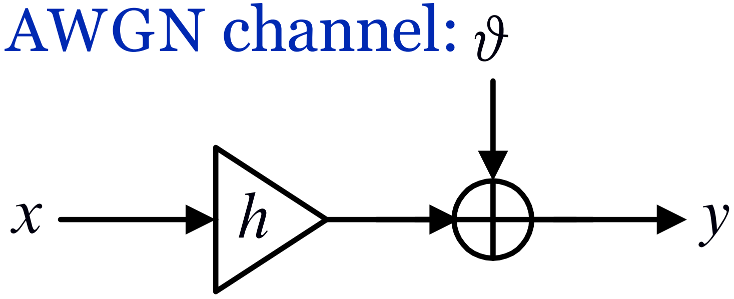


$h \in \mathbb{R}$: channel coefficient

$\vartheta \sim \mathcal{N}\left(0, \sigma_\vartheta^2\right)$

Learning an implication by example

- AWGN channel:



$h \in \mathbb{R}$: channel coefficient

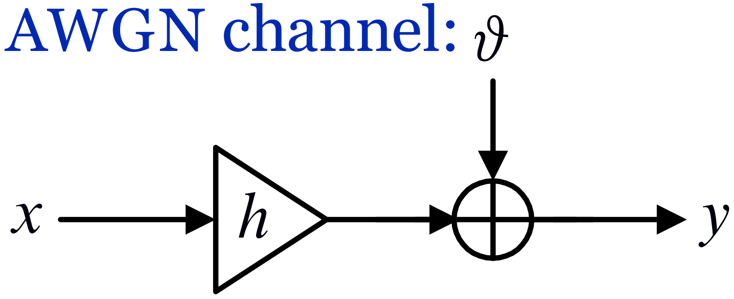
$\vartheta \sim \mathcal{N}\left(0, \sigma_\vartheta^2\right)$

- Signal to noise ratio:

$$\mathrm{SNR} = \frac{\mathrm{E}[\,|y|^2 \mid \vartheta = 0\,]}{\mathrm{E}[\,|y|^2 \mid x = 0\,]} = \frac{\mathrm{E}[\,|x|^2\,]}{\sigma_\vartheta^2} \cdot h^2$$

Learning an implication by example

- AWGN channel:



$h \in \mathbb{R}$: channel coefficient

$\vartheta \sim \mathcal{N}\left(0, \sigma_\vartheta^2\right)$

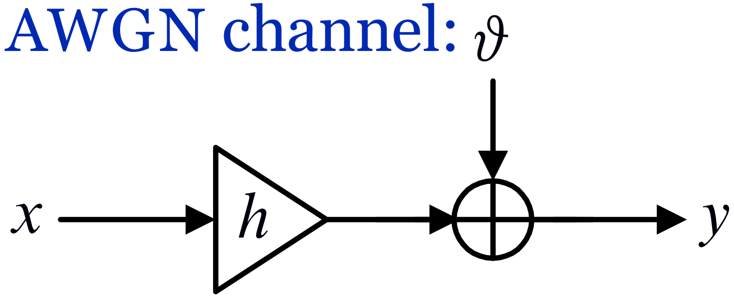
- Signal to noise ratio:

$$\mathrm{SNR} = \frac{\mathrm{E}[\,|y|^2 \mid \vartheta = 0\,]}{\mathrm{E}[\,|y|^2 \mid x = 0\,]} = \frac{\mathrm{E}[\,|x|^2\,]}{\sigma_\vartheta^2} \cdot h^2$$

- A question: $(h \leftarrow h/2) \longrightarrow (\mathrm{SNR} \leftarrow$

Learning an implication by example

- AWGN channel:



$h \in \mathbb{R}$: channel coefficient

$\vartheta \sim \mathcal{N}\left(0, \sigma_\vartheta^2\right)$

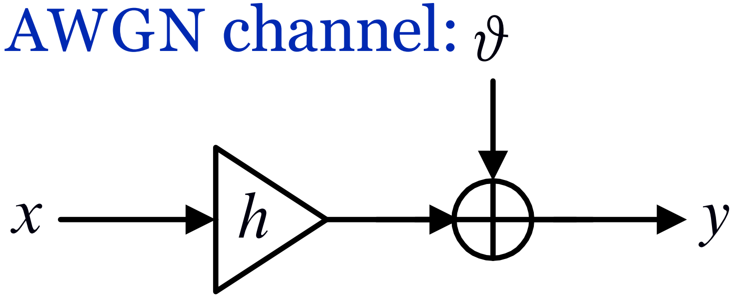
- Signal to noise ratio:

$$\mathrm{SNR} = \frac{\mathrm{E}[\,|y|^2 \mid \vartheta = 0\,]}{\mathrm{E}[\,|y|^2 \mid x = 0\,]} = \frac{\mathrm{E}[\,|x|^2\,]}{\sigma_\vartheta^2} \cdot h^2$$

- A question: $\left(h \leftarrow h/2\right) \longrightarrow \left(\mathrm{SNR} \leftarrow \mathrm{SNR}/4\right)$?

Learning an implication by example

- **AWGN channel:**



$h \in \mathbb{R}$: channel coefficient

$\vartheta \sim \mathcal{N}\left(0, \sigma_\vartheta^2\right)$

- **Signal to noise ratio:**

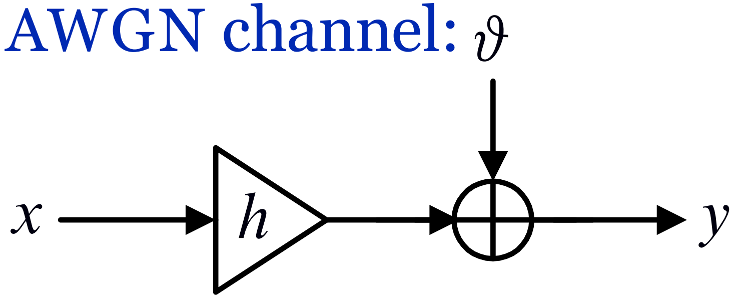
$$\mathrm{SNR} = \frac{\mathrm{E}[\,|y|^2 \mid \vartheta = 0\,]}{\mathrm{E}[\,|y|^2 \mid x = 0\,]} = \frac{\mathrm{E}[\,|x|^2\,]}{\sigma_\vartheta^2} \cdot h^2$$

- **A question:** $\left(h \leftarrow h/2\right) \longrightarrow \left(\mathrm{SNR} \leftarrow \mathrm{SNR}/4\right)$?

$\longrightarrow$ *That's only right if σ_ϑ^2 is independent of h.*

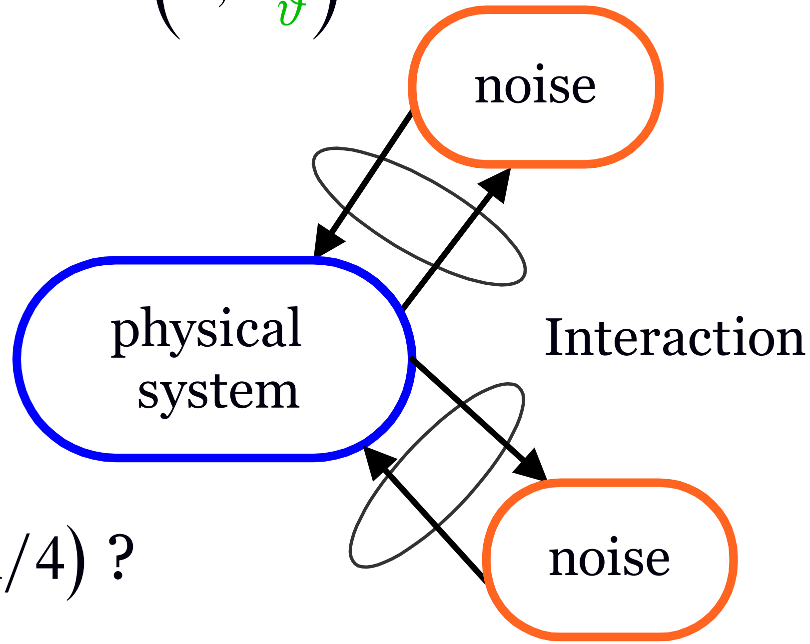
Learning an implication by example

- AWGN channel:



$h \in \mathbb{R}$: channel coefficient

$\vartheta \sim \mathcal{N}\left(0, \sigma_\vartheta^2\right)$



- Signal to noise ratio:

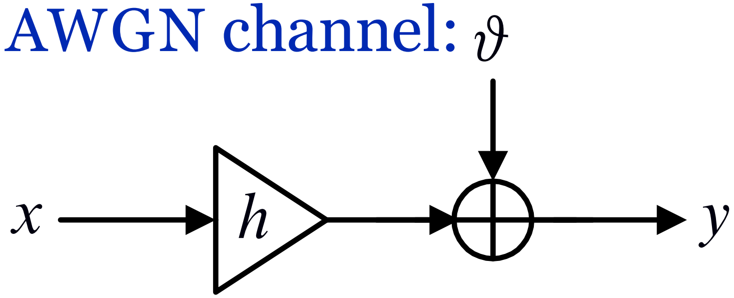
$$\mathrm{SNR} = \frac{\mathrm{E}[\,|y|^2 \mid \vartheta = 0\,]}{\mathrm{E}[\,|y|^2 \mid x = 0\,]} = \frac{\mathrm{E}[\,|x|^2\,]}{\sigma_\vartheta^2} \cdot h^2$$

- A question: $\left(h \leftarrow h/2\right) \longrightarrow \left(\mathrm{SNR} \leftarrow \mathrm{SNR}/4\right)$?

$\longrightarrow$ *That's only right if σ_ϑ^2 is independent of h.*

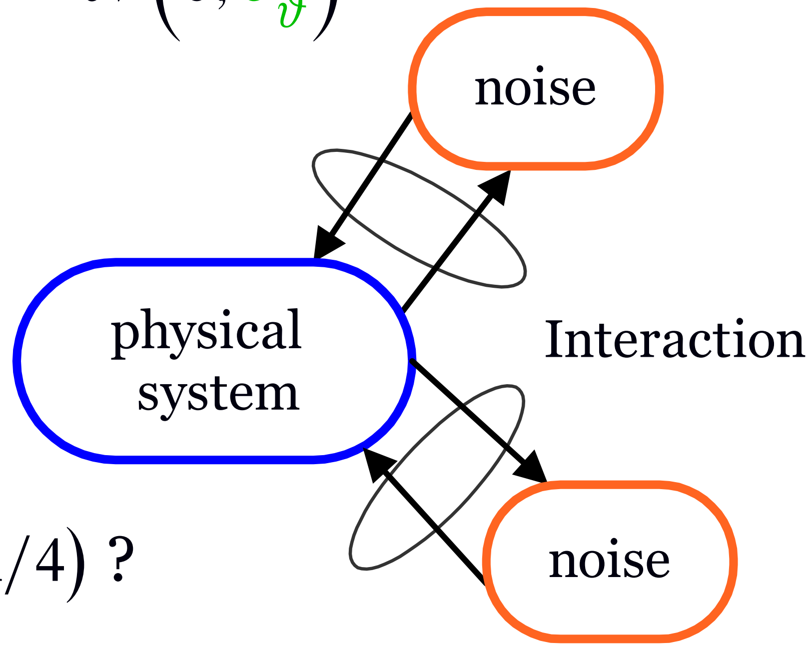
Learning an implication by example

- AWGN channel: ϑ



$h \in \mathbb{R}$: channel coefficient

$\vartheta \sim \mathcal{N}\left(0, \sigma_\vartheta^2\right)$



- Signal to noise ratio:

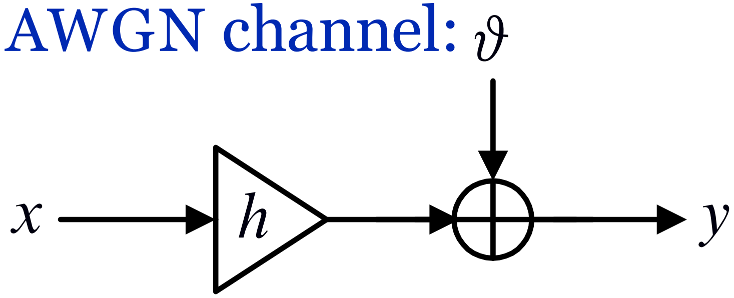
$$\mathrm{SNR} = \frac{\mathrm{E}[\,|y|^2 \mid \vartheta = 0\,]}{\mathrm{E}[\,|y|^2 \mid x = 0\,]} = \frac{\mathrm{E}[\,|x|^2\,]}{\sigma_\vartheta^2} \cdot h^2$$

- A question: $(h \leftarrow h/2) \longrightarrow (\mathrm{SNR} \leftarrow \mathrm{SNR}/4)$?

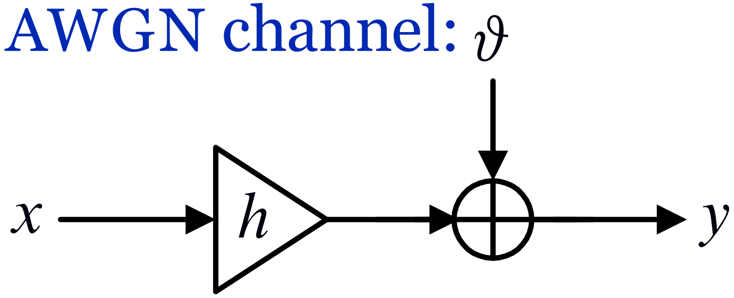
$\longrightarrow$ *That's only right if σ_ϑ^2 is independent of h.*

Consistent modeling of a physical system as an AWGN–channel may enforce dependencies between h und σ_ϑ^2.

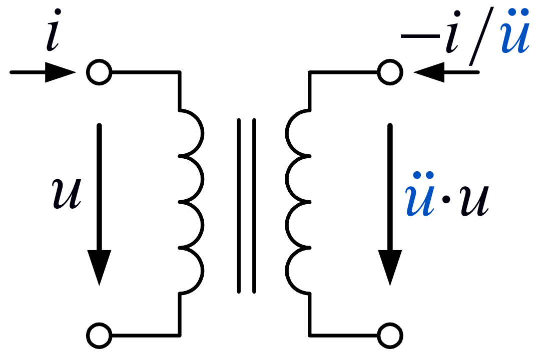
- AWGN channel:



- AWGN channel:



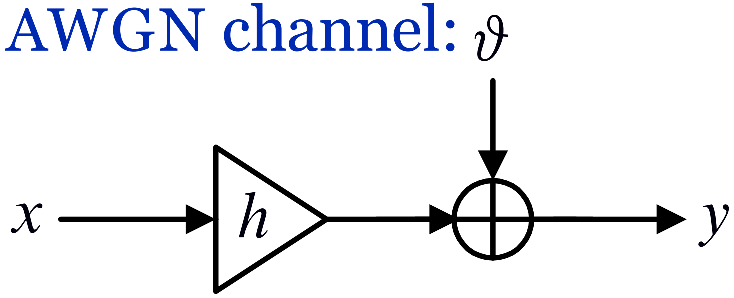
- Physical communication system:



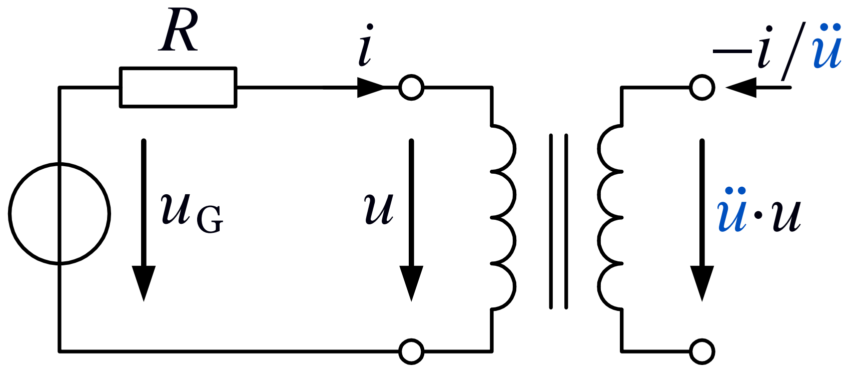
$\ddot{u} \in \mathbb{R}$: Transformation ratio

Learning an implication by example

- AWGN channel:

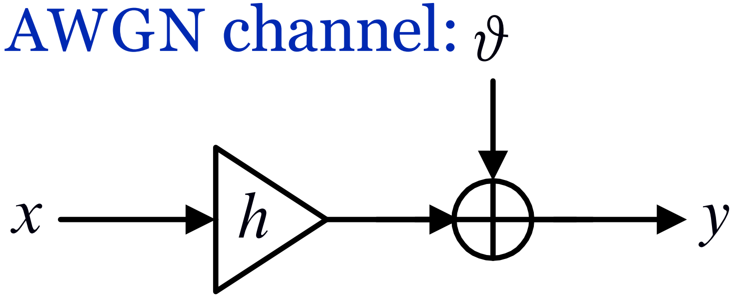


- Physical communication system:

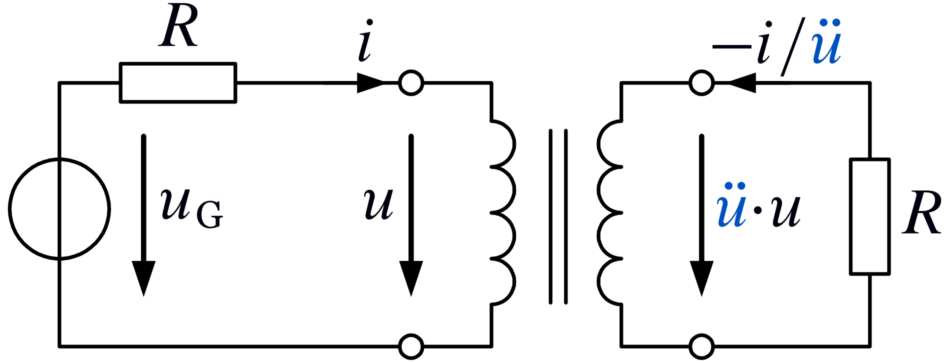


$\ddot{u} \in \mathbb{R}$: Transformation ratio

* AWGN channel:

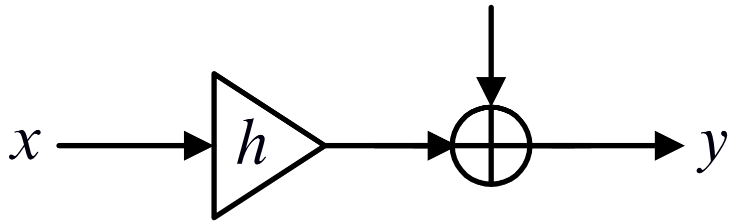


* Physical communication system:

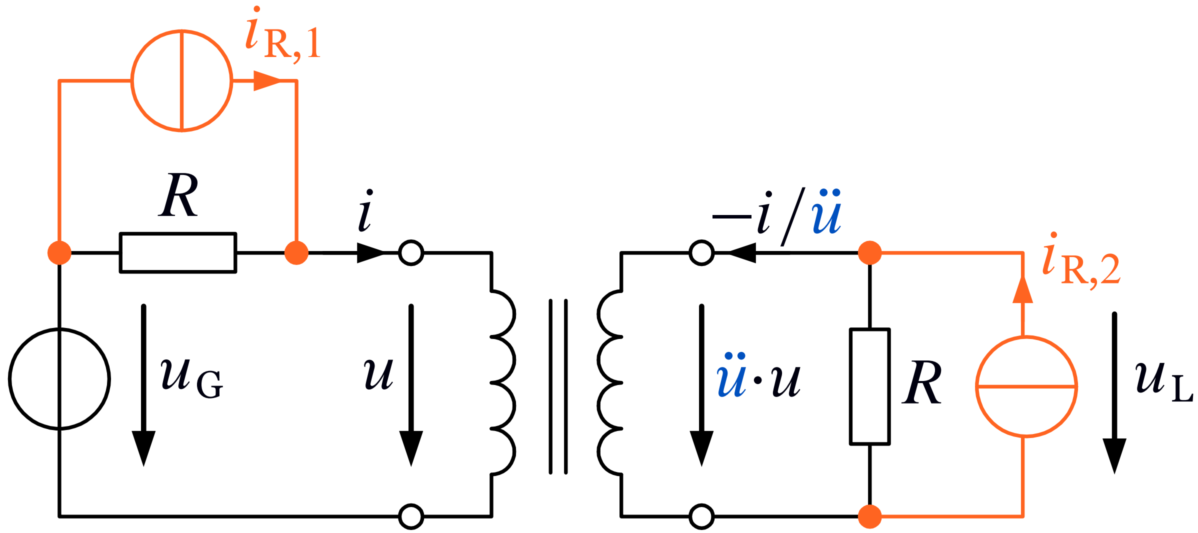


$\ddot{u} \in \mathbb{R}$: Transformation ratio

- AWGN channel:



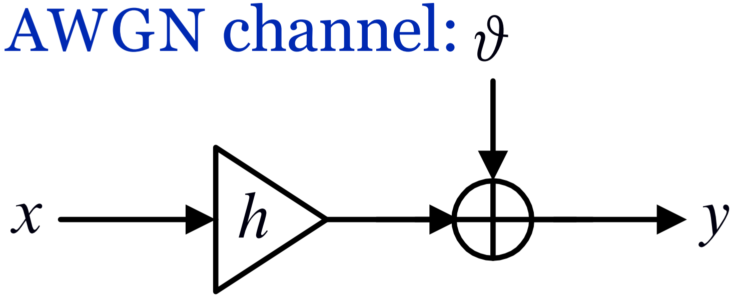
- Physical communication system:



$\ddot{u} \in \mathbb{R}$: Transformation ratio

Learning an implication by example

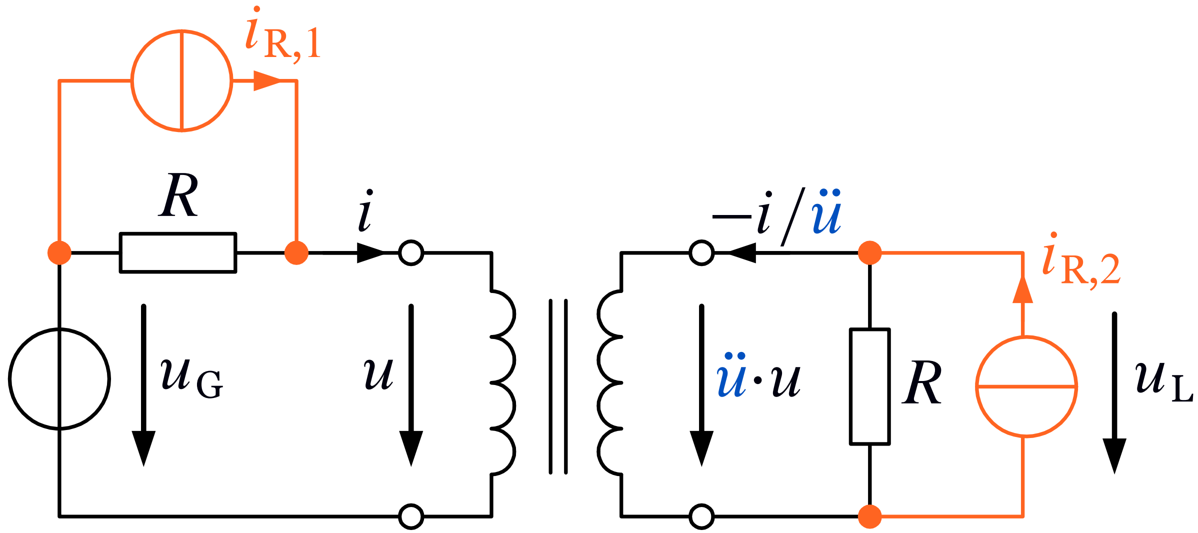
- AWGN channel:



- Signal assignment:

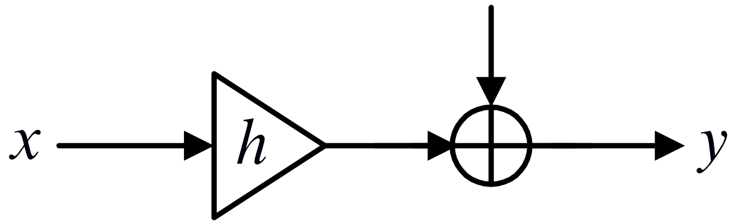
$$x = u_G, \quad y = u_L$$

- Physical communication system:



$\ddot{u} \in \mathbb{R}$: Transformation ratio

- AWGN channel: ϑ

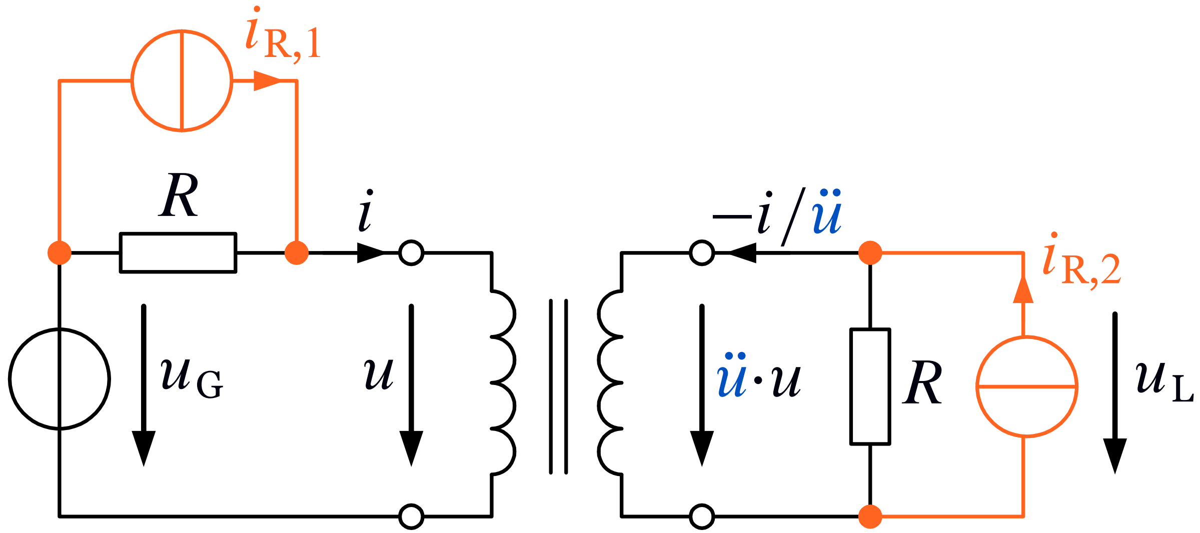


- Signal assignment:

$$x = u_\mathrm{G}, \quad y = u_\mathrm{L}$$

- Implications:

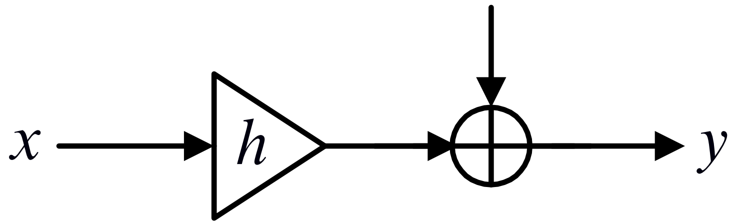
- Physical communication system:



$\ddot{u} \in \mathbb{R}$: Transformation ratio

Learning an implication by example

- AWGN channel: ϑ

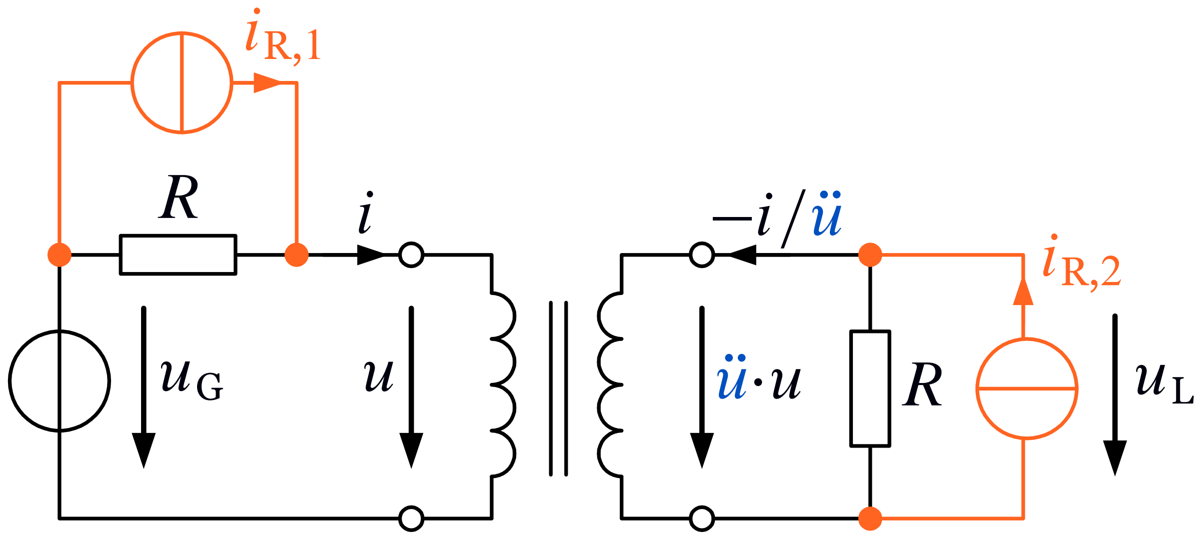


- Signal assignment:

$$x = u_{\mathrm{G}}, \quad y = u_{\mathrm{L}}$$

- Implications:

- Physical communication system:



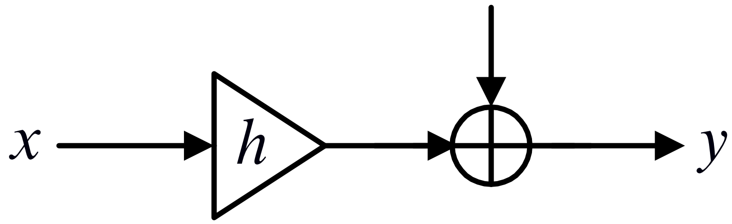
$\ddot{u} \in \mathbb{R}$: Transformation ratio

- From circuit analysis:

$$u_{\mathrm{L}} = \frac{\ddot{u}}{1+\ddot{u}^2}\, u_{\mathrm{G}} + R\ddot{u}\, \frac{i_{\mathrm{R},1} + \ddot{u}\, i_{\mathrm{R},2}}{1+\ddot{u}^2}$$

Learning an implication by example

- AWGN channel: ϑ

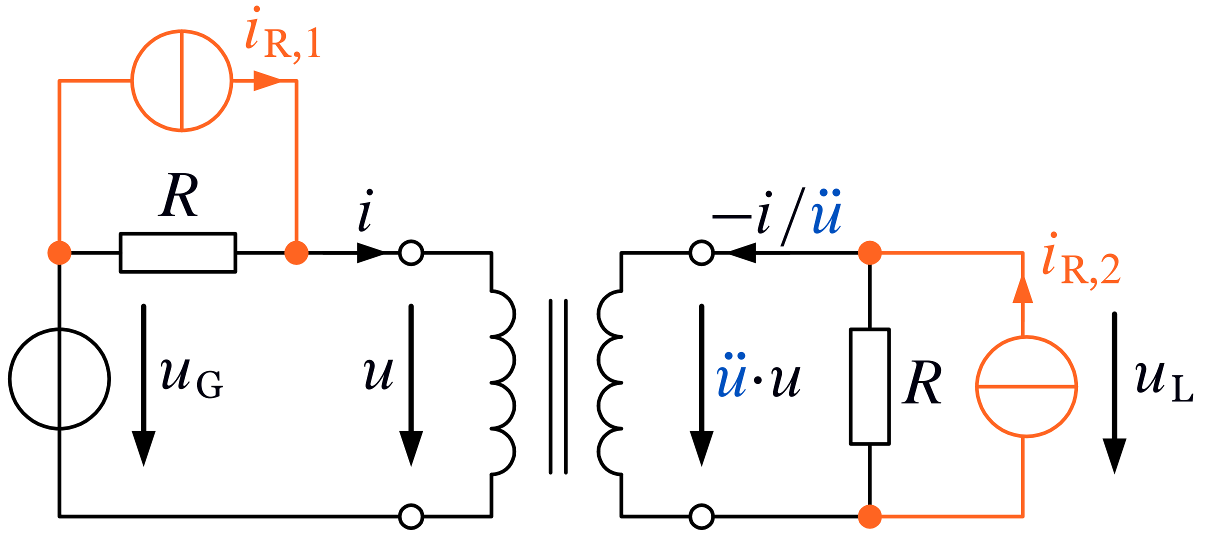


- Signal assignment:

$$x = u_{\mathrm{G}}, \qquad y = u_{\mathrm{L}}$$

- Implications:

- Physical communication system:

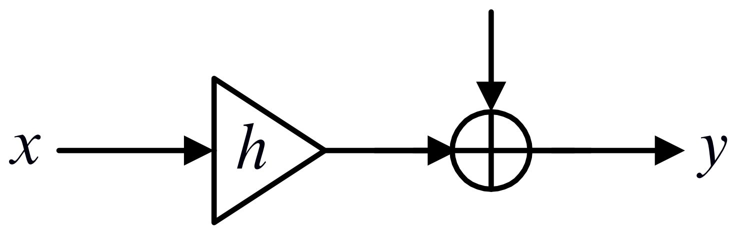


$\ddot{u} \in \mathbb{R}$: Transformation ratio

- From circuit analysis:

$$u_{\mathrm{L}} = \underbrace{\frac{\ddot{u}}{1+\ddot{u}^2}}_{h} u_{\mathrm{G}} + \underbrace{R\ddot{u}\,\frac{i_{\mathrm{R},1} + \ddot{u}\, i_{\mathrm{R},2}}{1+\ddot{u}^2}}_{\vartheta}$$

- **AWGN channel:** ϑ



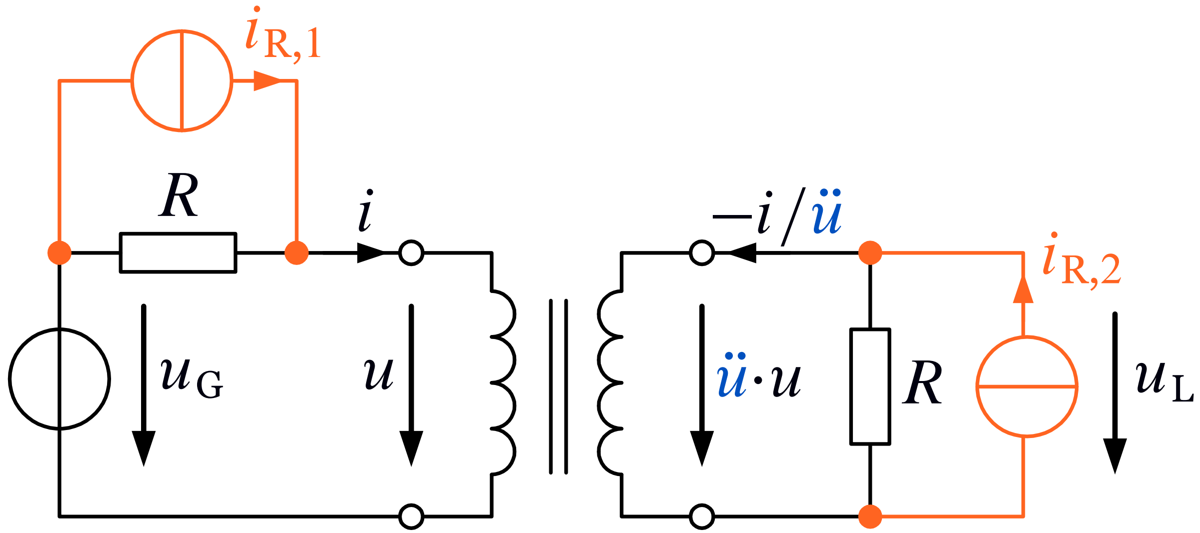
- **Signal assignment:**

$$x = u_{\mathrm{G}}, \quad y = u_{\mathrm{L}}$$

- **Implications:**

$$h = \frac{\ddot{u}}{1 + \ddot{u}^2}, \quad |h| \le 1/2$$

- **Physical communication system:**



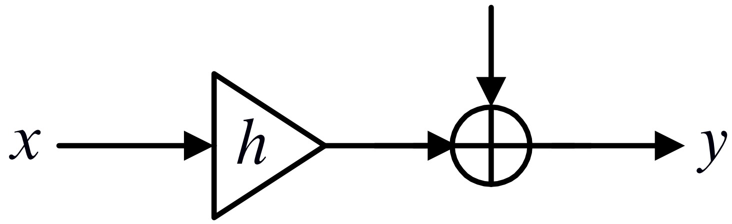
$\ddot{u} \in \mathbb{R}$: Transformation ratio

- **From circuit analysis:**

$$u_{\mathrm{L}} = \underbrace{\frac{\ddot{u}}{1 + \ddot{u}^2}}_{h} u_{\mathrm{G}} + \underbrace{R\ddot{u}\,\frac{i_{\mathrm{R},1} + \ddot{u}\, i_{\mathrm{R},2}}{1 + \ddot{u}^2}}_{\vartheta}$$

Learning an implication by example

- AWGN channel: ϑ



- Signal assignment:

$$x = u_\mathrm{G}, \quad y = u_\mathrm{L}$$

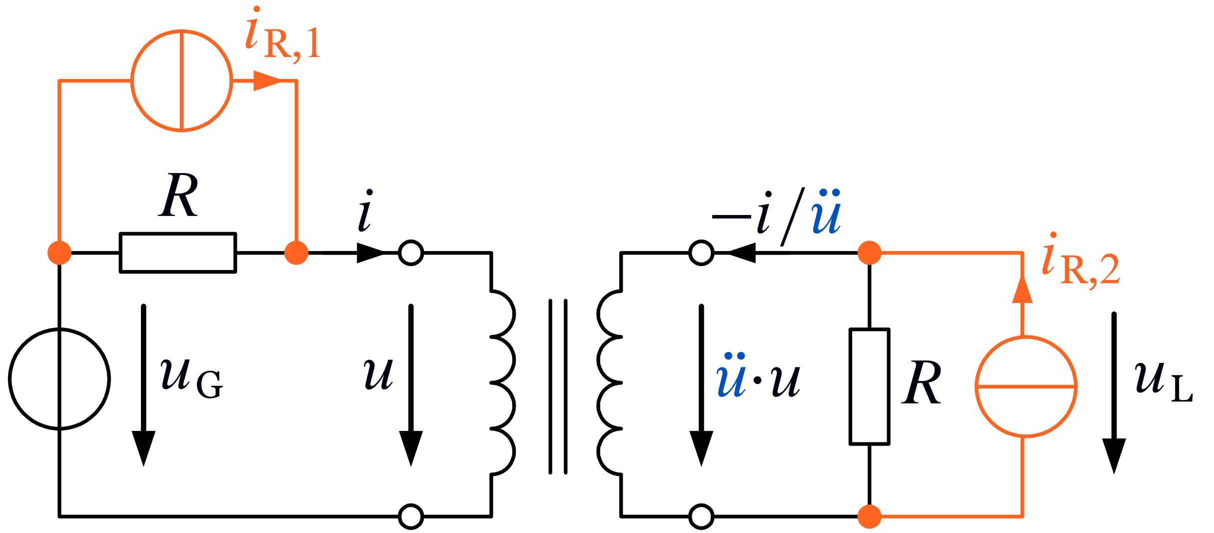
- Implications:

$$h = \frac{\ddot{u}}{1+\ddot{u}^2}, \quad |h| \leq 1/2$$

$$\sigma_\vartheta^2 = \sigma_0^2 \left(1 - \sqrt{1-4h^2}\right)$$

$$\sigma_0^2 = 2kTBR$$

- Physical communication system:



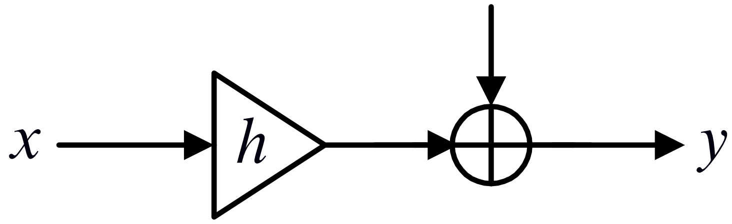
$\ddot{u} \in \mathbb{R}$: Transformation ratio

- From circuit analysis:

$$u_\mathrm{L} = \underbrace{\frac{\ddot{u}}{1+\ddot{u}^2}}_{h} u_\mathrm{G} + \underbrace{R\ddot{u}\,\frac{i_{\mathrm{R},1} + \ddot{u}\, i_{\mathrm{R},2}}{1+\ddot{u}^2}}_{\vartheta}$$

Learning an implication by example

- AWGN channel: ϑ



- Signal assignment:

$$x = u_\mathrm{G}, \quad y = u_\mathrm{L}$$

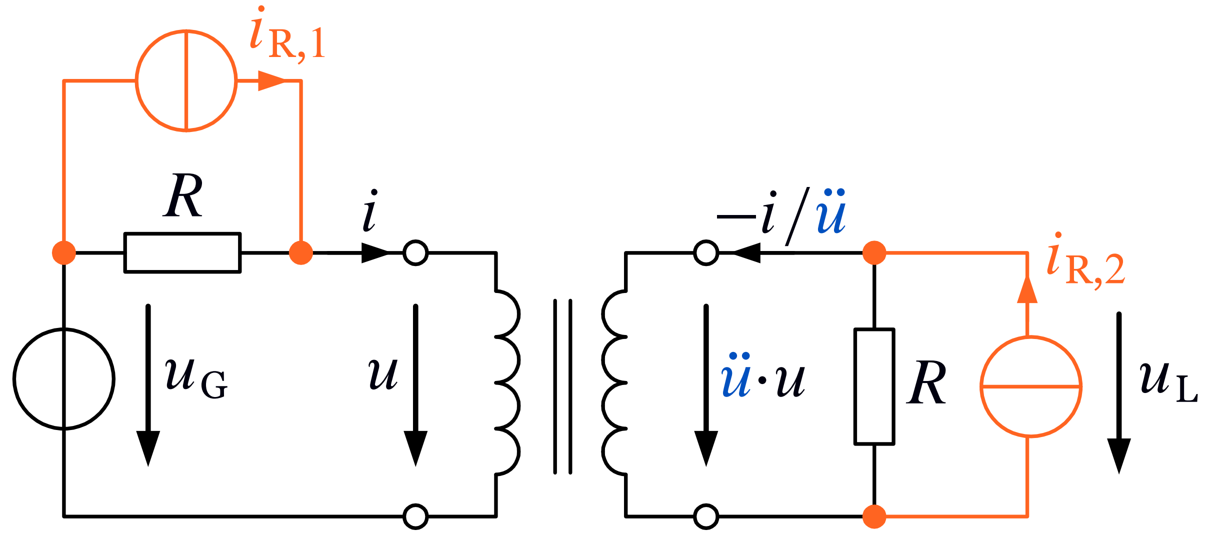
- Implications:

$$h = \frac{\ddot{u}}{1+\ddot{u}^2}, \quad |h| \leq 1/2$$

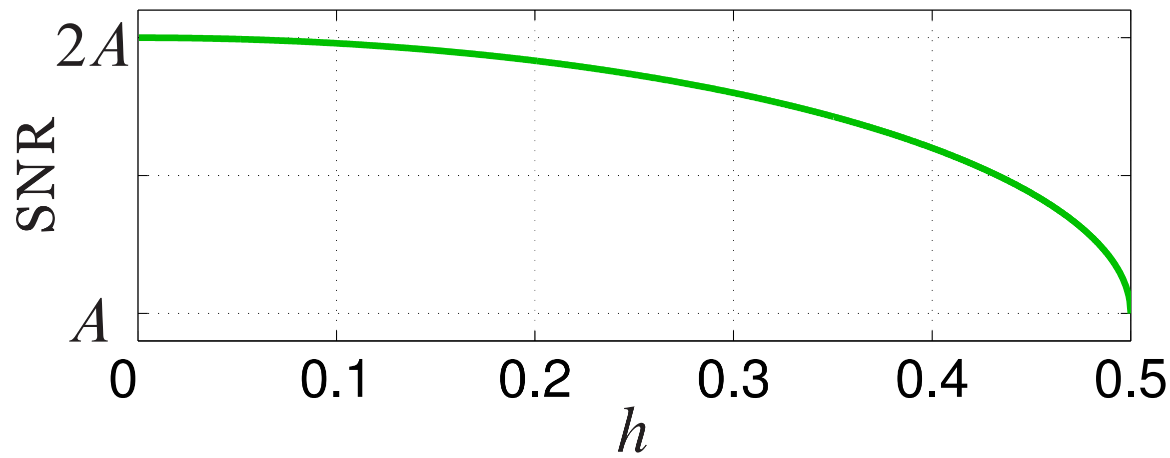
$$\sigma_\vartheta^2 = \sigma_0^2 \left(1 - \sqrt{1-4h^2}\,\right)$$

$$\sigma_0^2 = 2kTBR$$

- Physical communication system:



- Signal to noise ratio:



Modeling linear systems consistent with physics

* How do we *systematically* find all relevant implications?

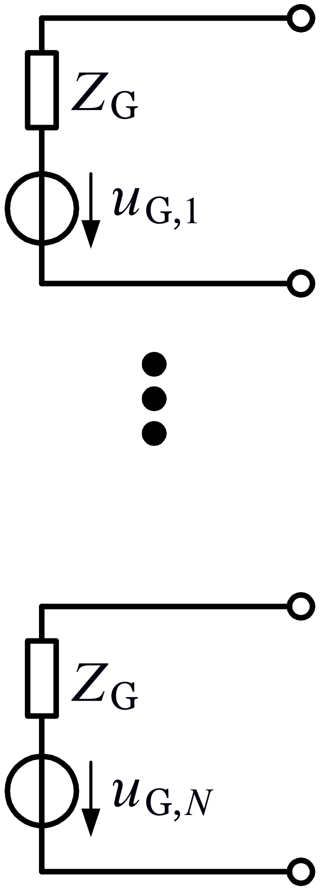
1. To Each input and each output of the system we assign a *port* of a linear *multiport*.
2. Each port is described by *two* conjugated physical quantities. We employ electric *current* und *voltage*.
3. The relationships of the port variables are deduced from the *laws of Nature*, and the *physical/technological* constraints.
4. Signals are *assigned* to physical port variables.
5. All *implications* for the system function are systematically determined by *multiport analysis*.

Components of a multiport model

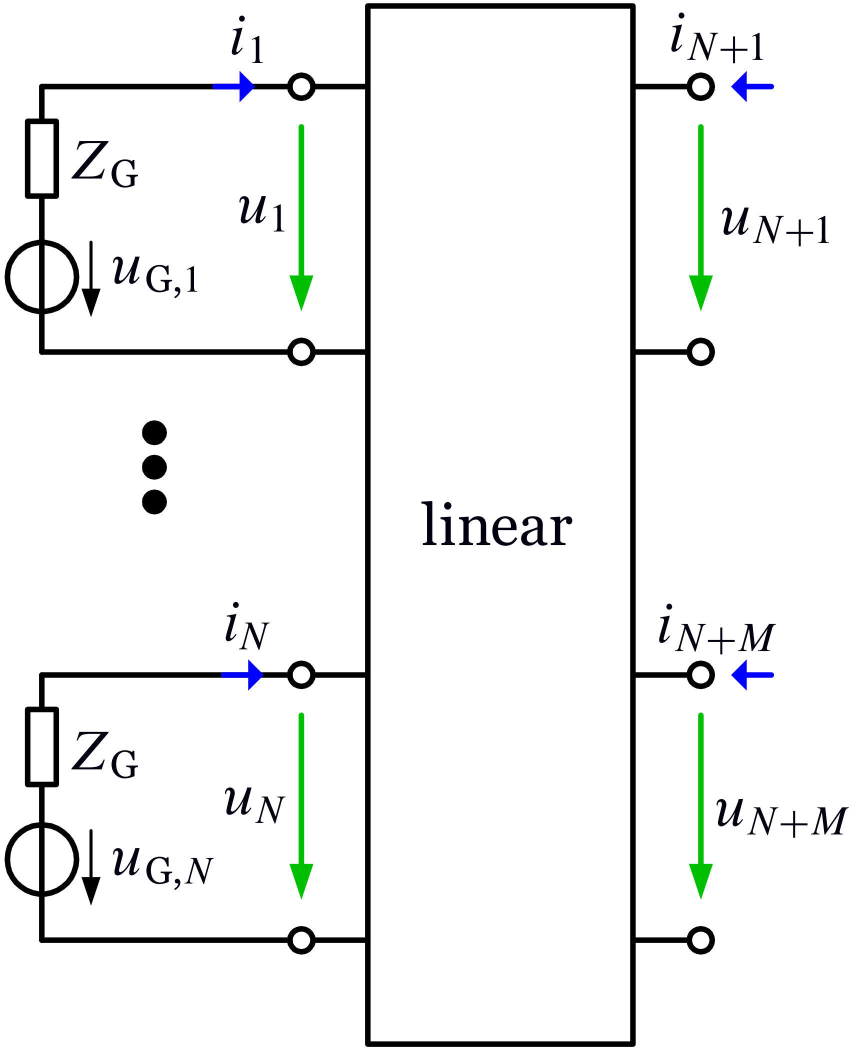
* The multiport description of a communication system comprises:

1. Generation and injection of the input signals.
2. Description of the coupling between all ports.
3. Properties and coupling of physical noise sources.
4. Extraction and observation of the output signals.

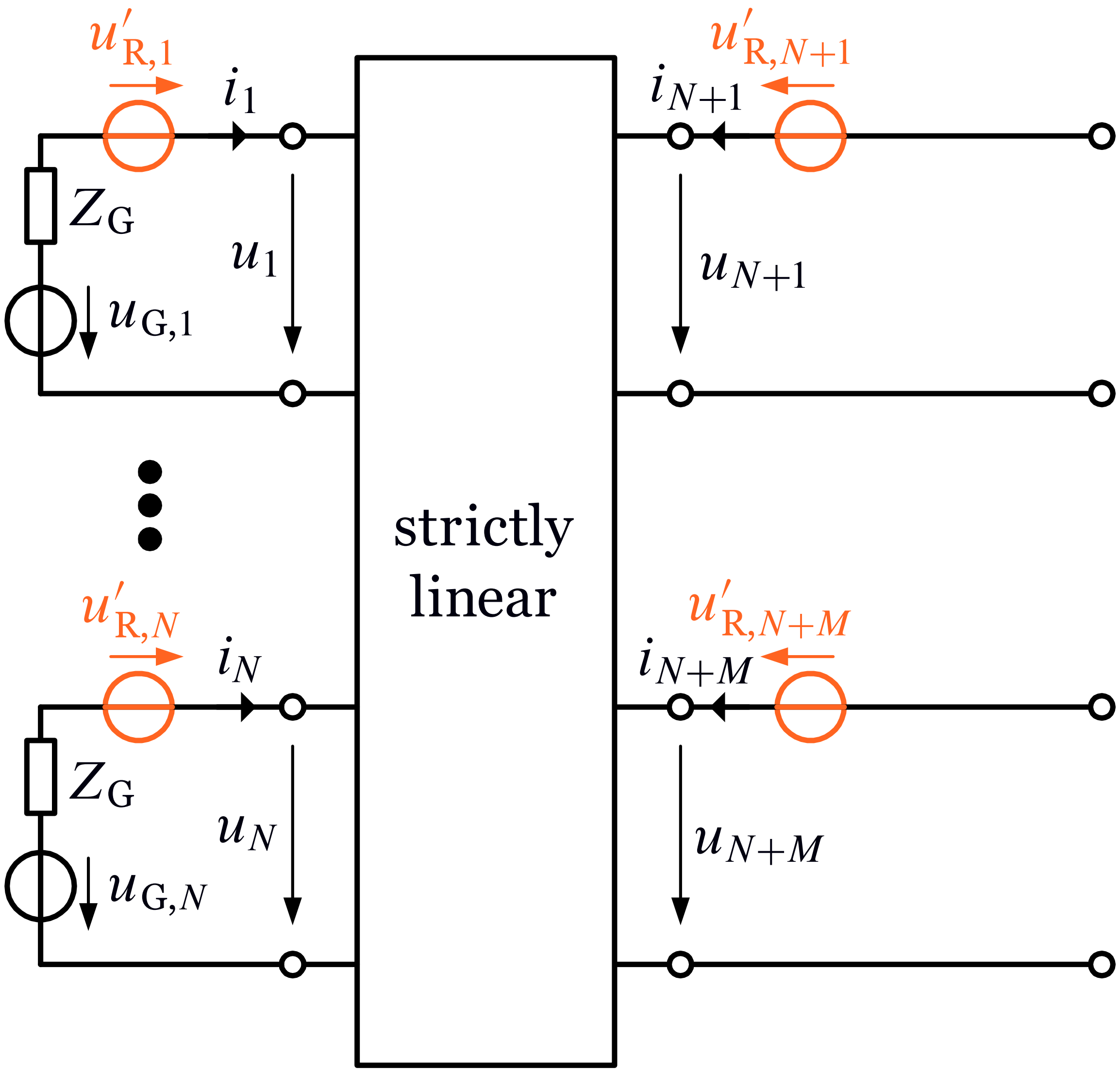
- Signal generation:



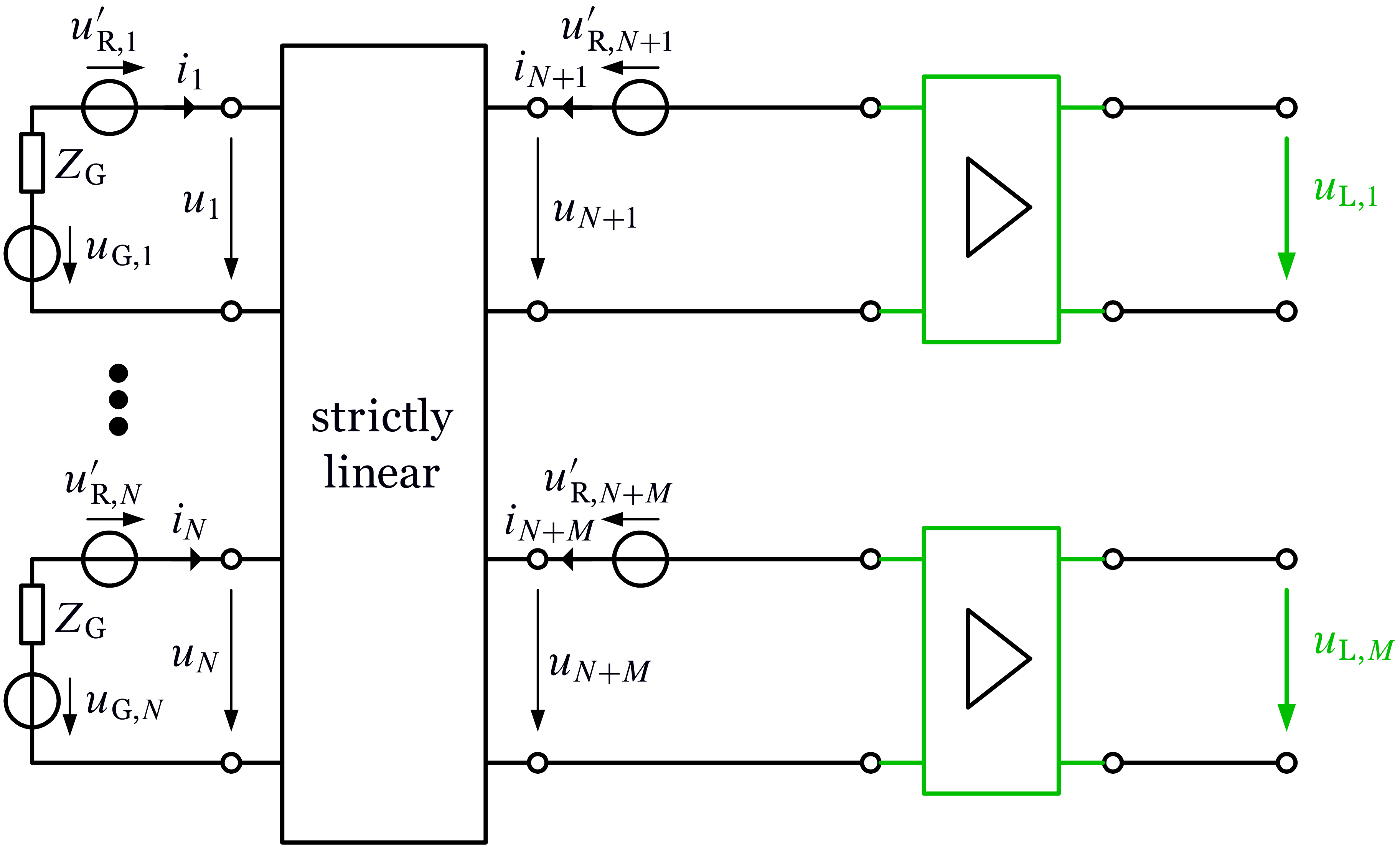
* Communication multiport with $(N+M)$ ports



* *Noisy* communication multiport

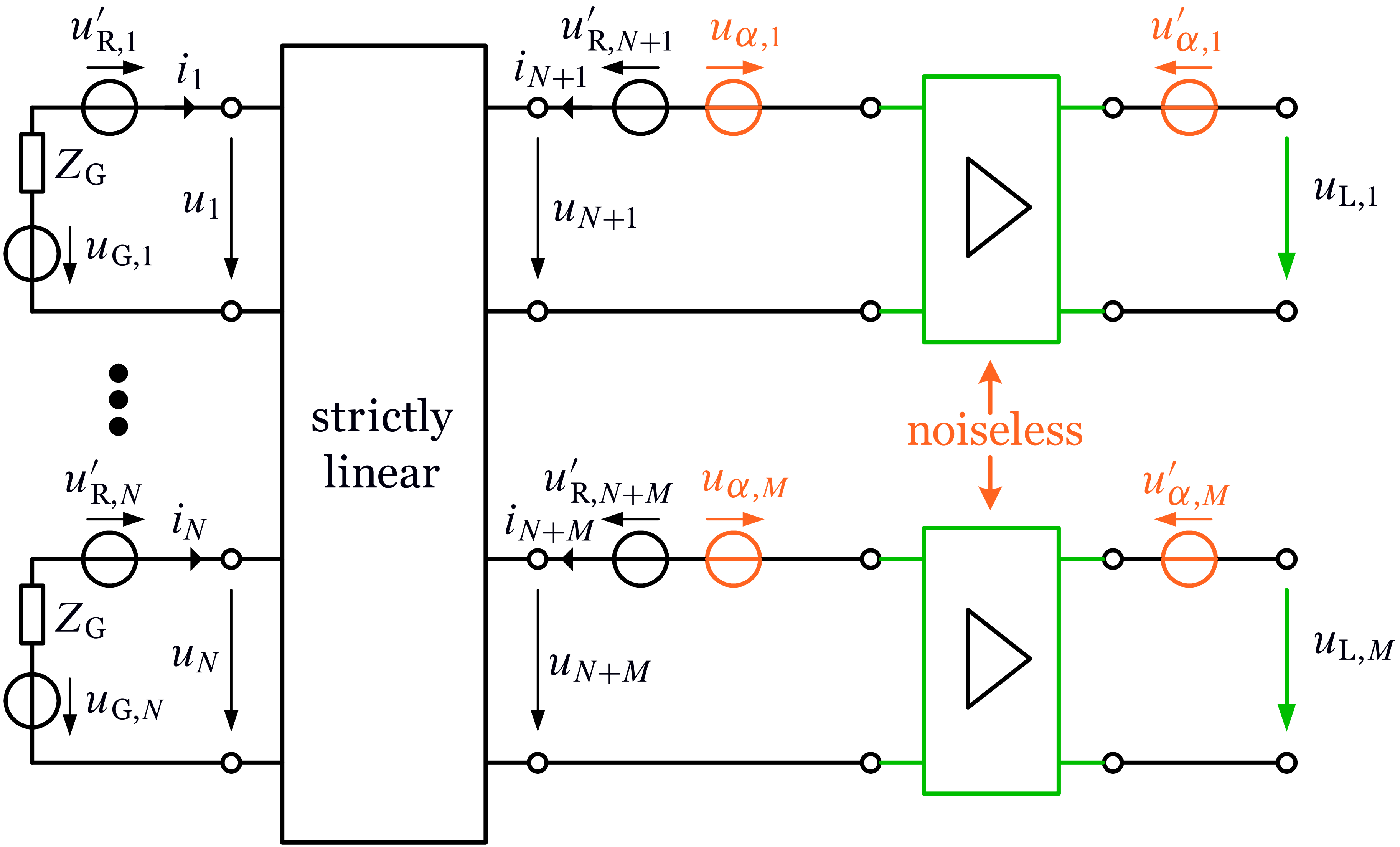


- Receive signal amplification and observation:

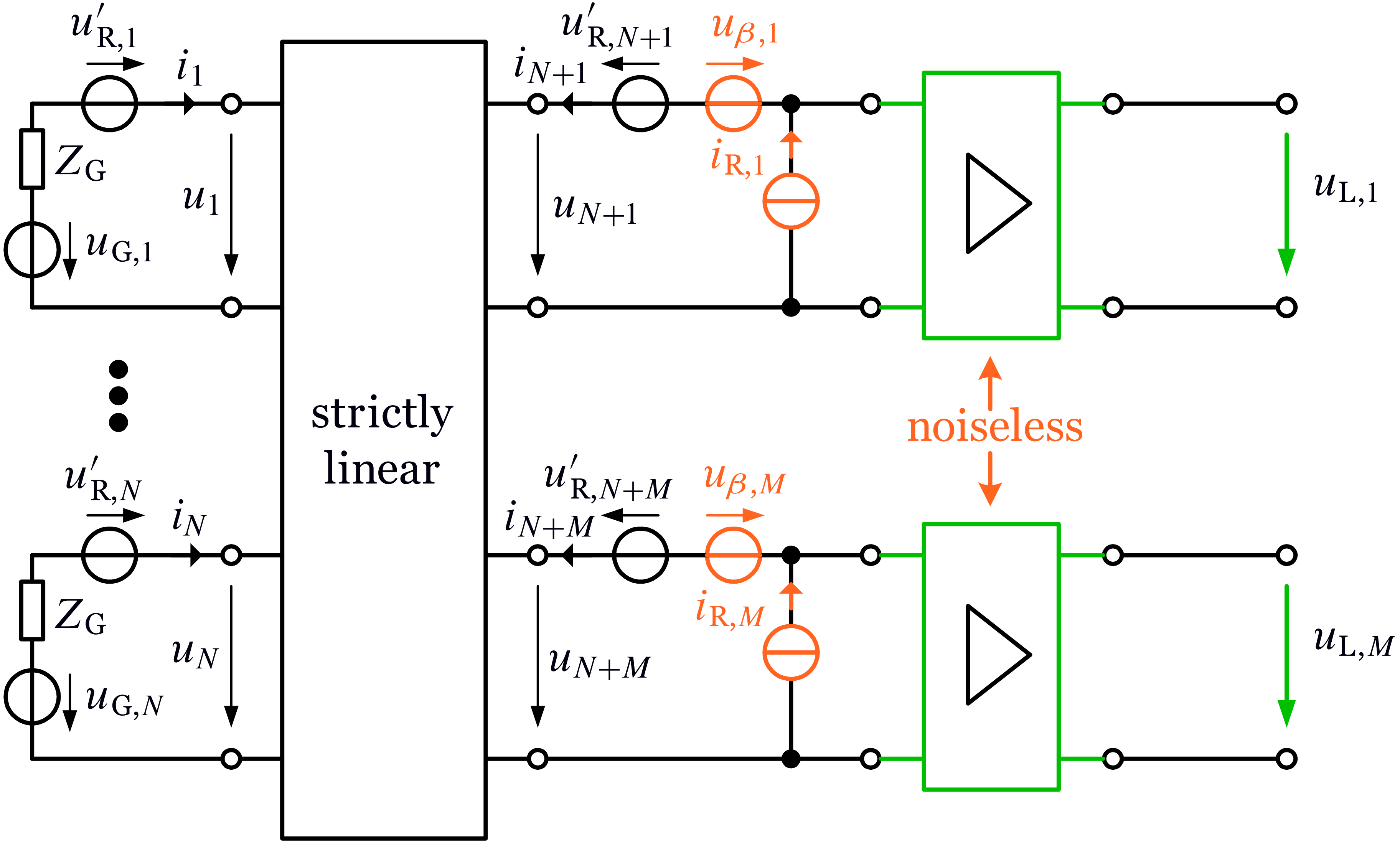


Linear multiport model of communication systems

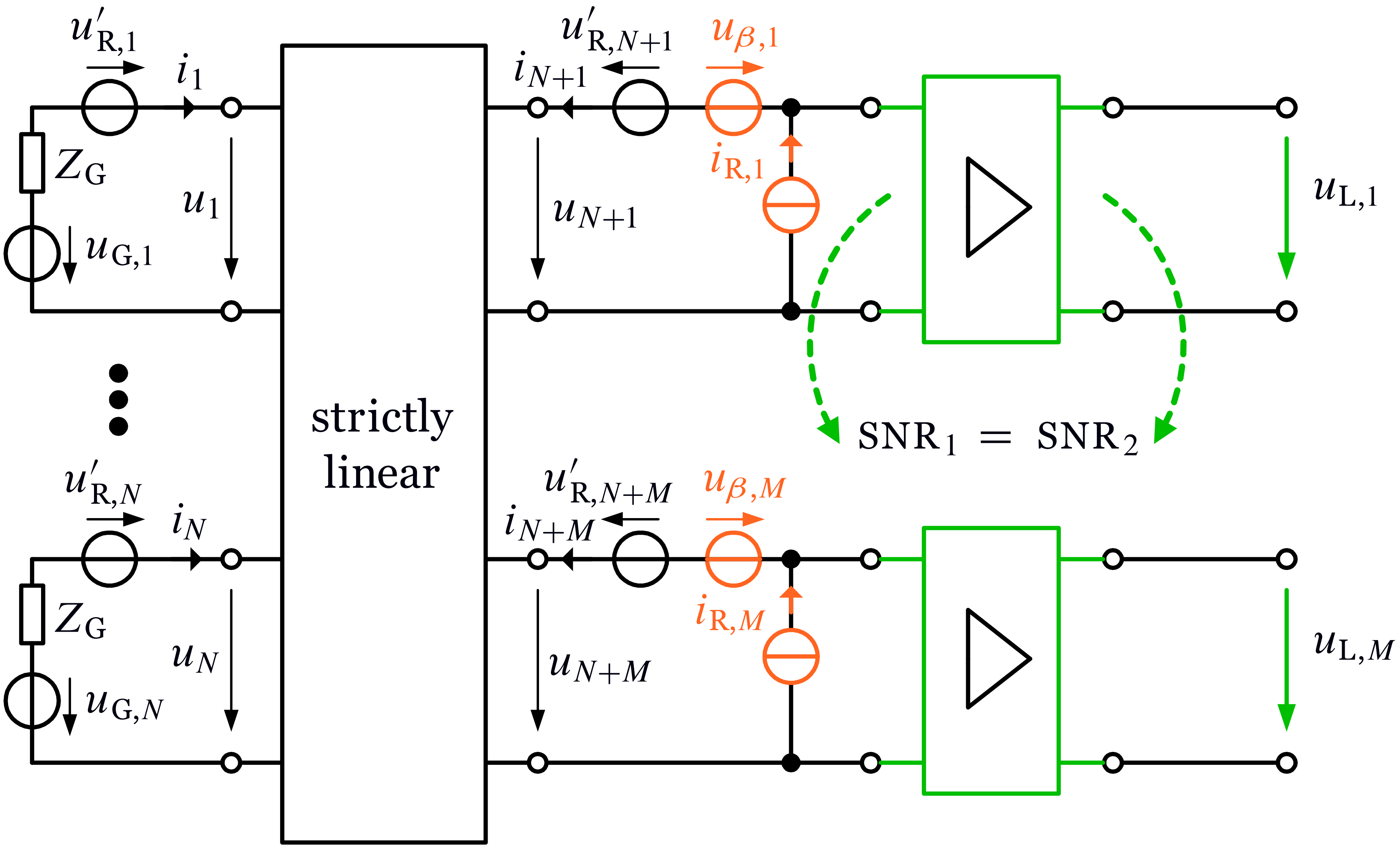
* *Noisy* receive amplifier



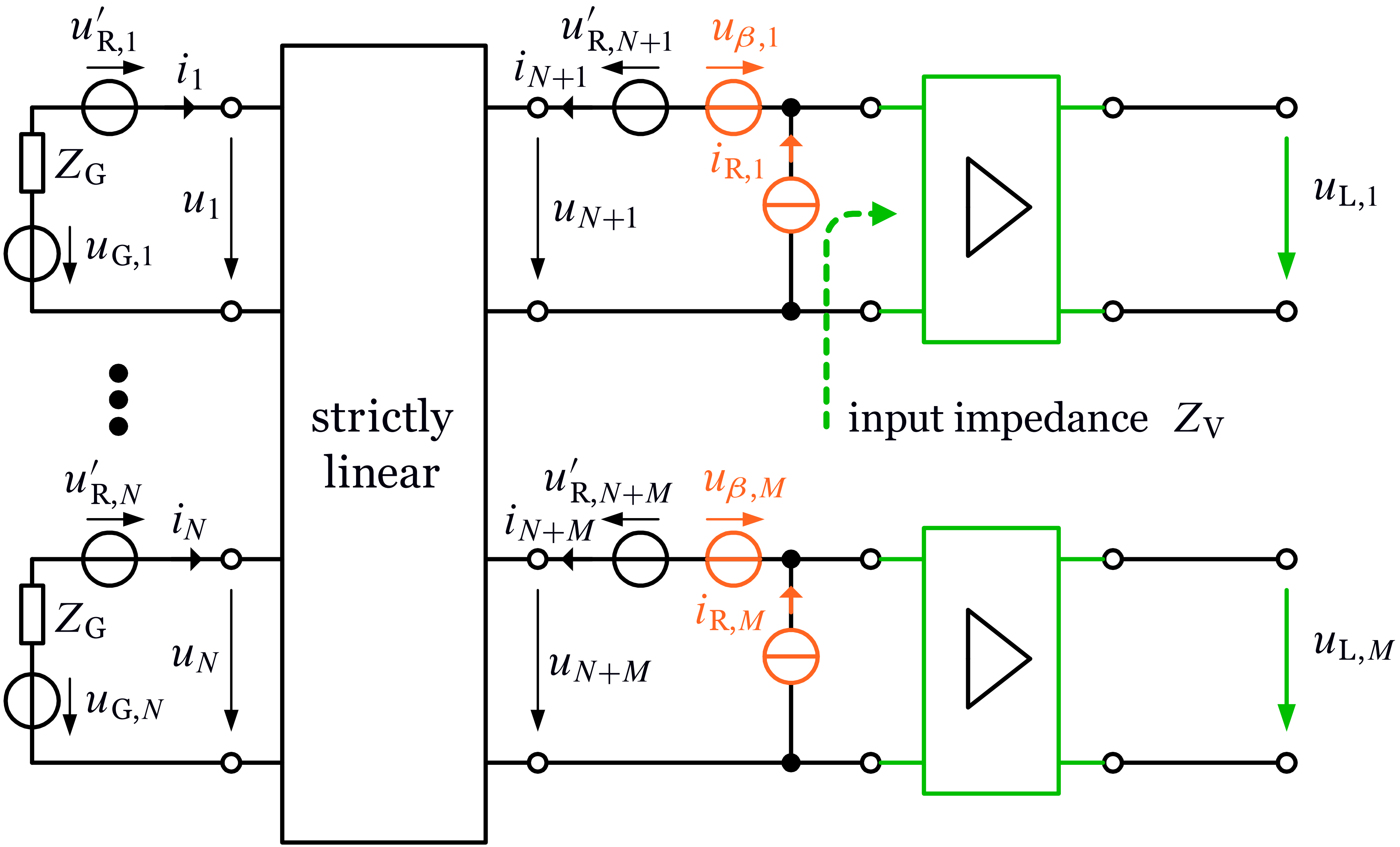
✻ Modeling the effects of noise at the amplifier's input



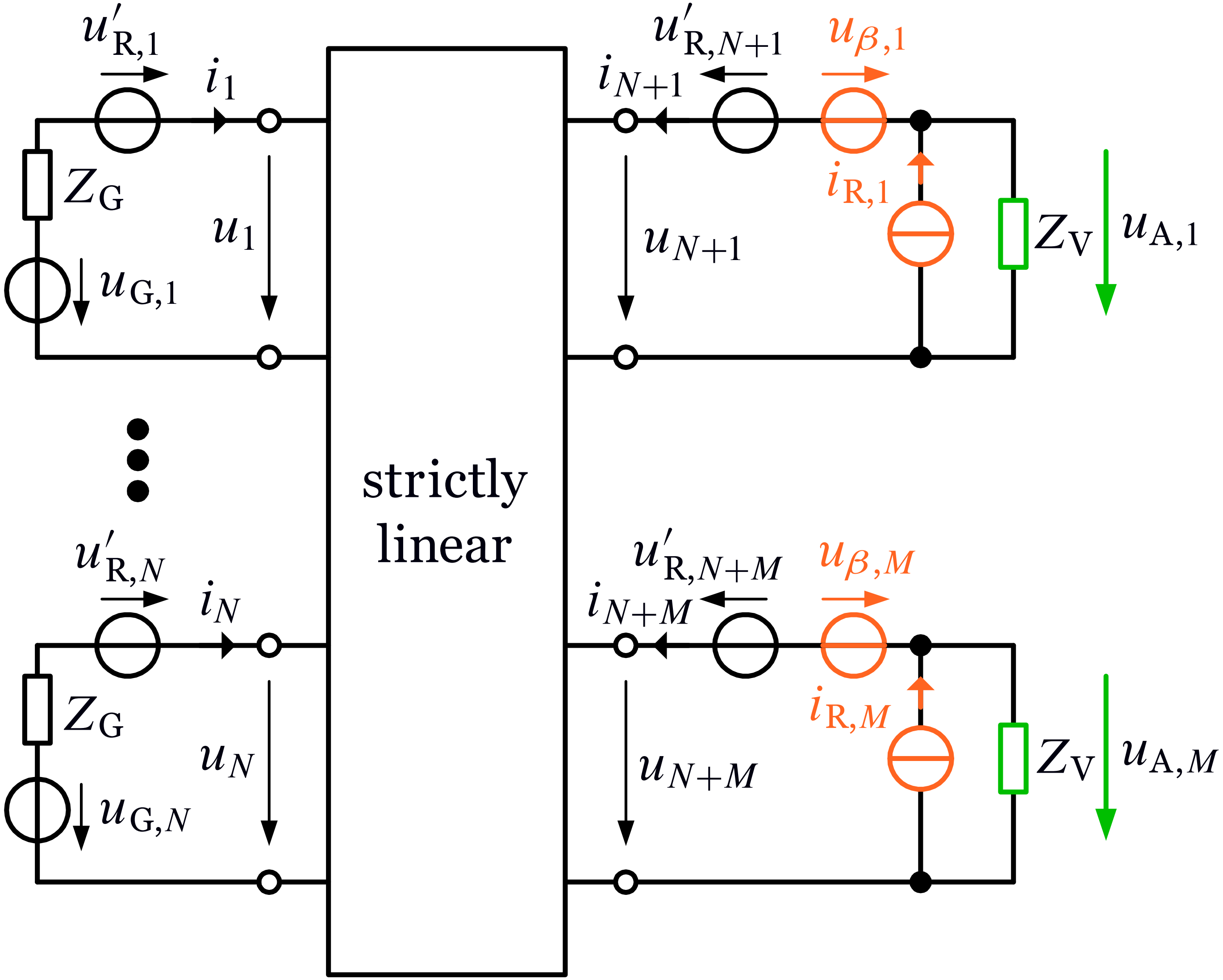
Modeling the effects of noise at the amplifier's input



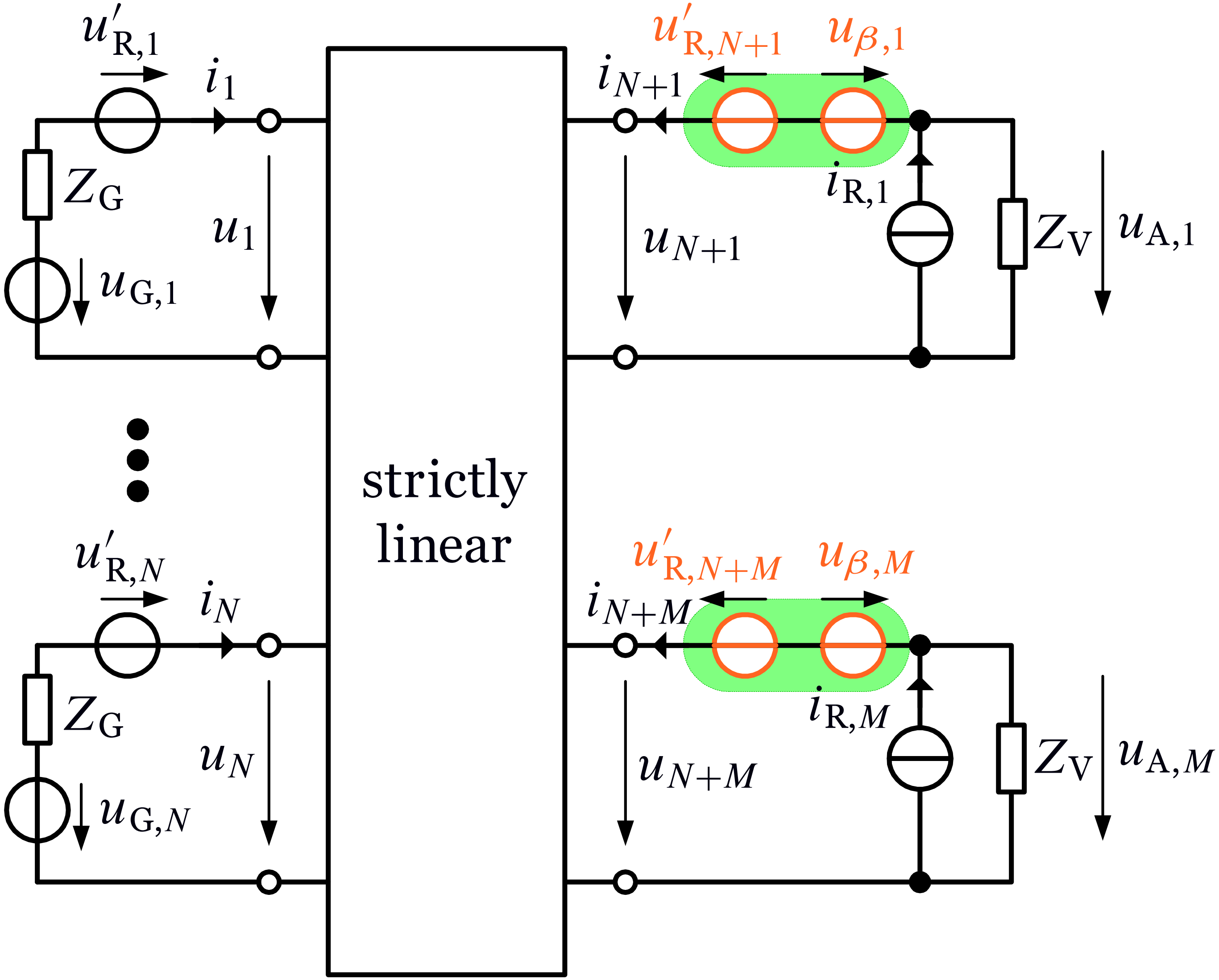
✻ Modeling the effects of noise at the amplifier's input



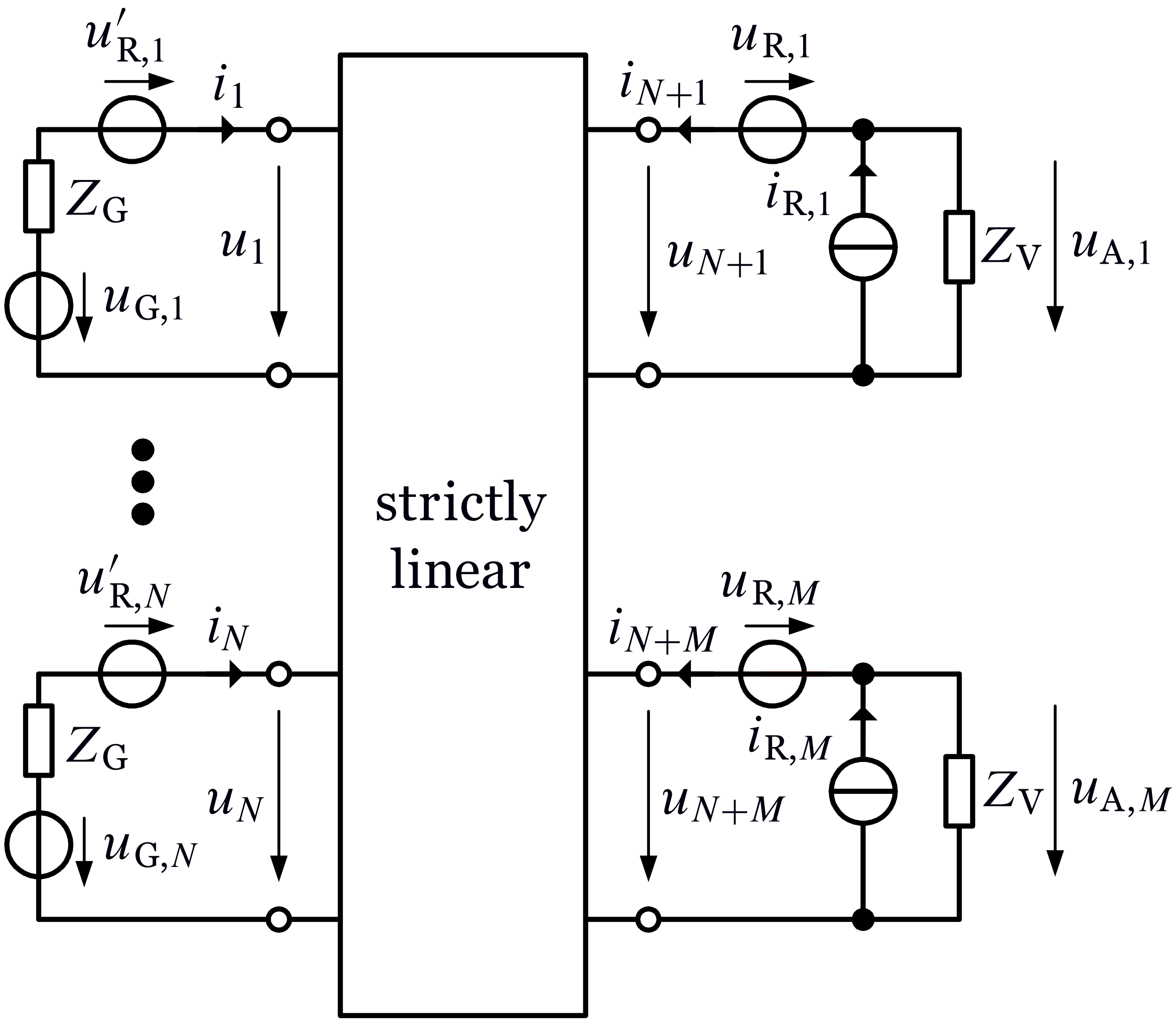
✻ Modeling the effects of noise at the amplifier's input



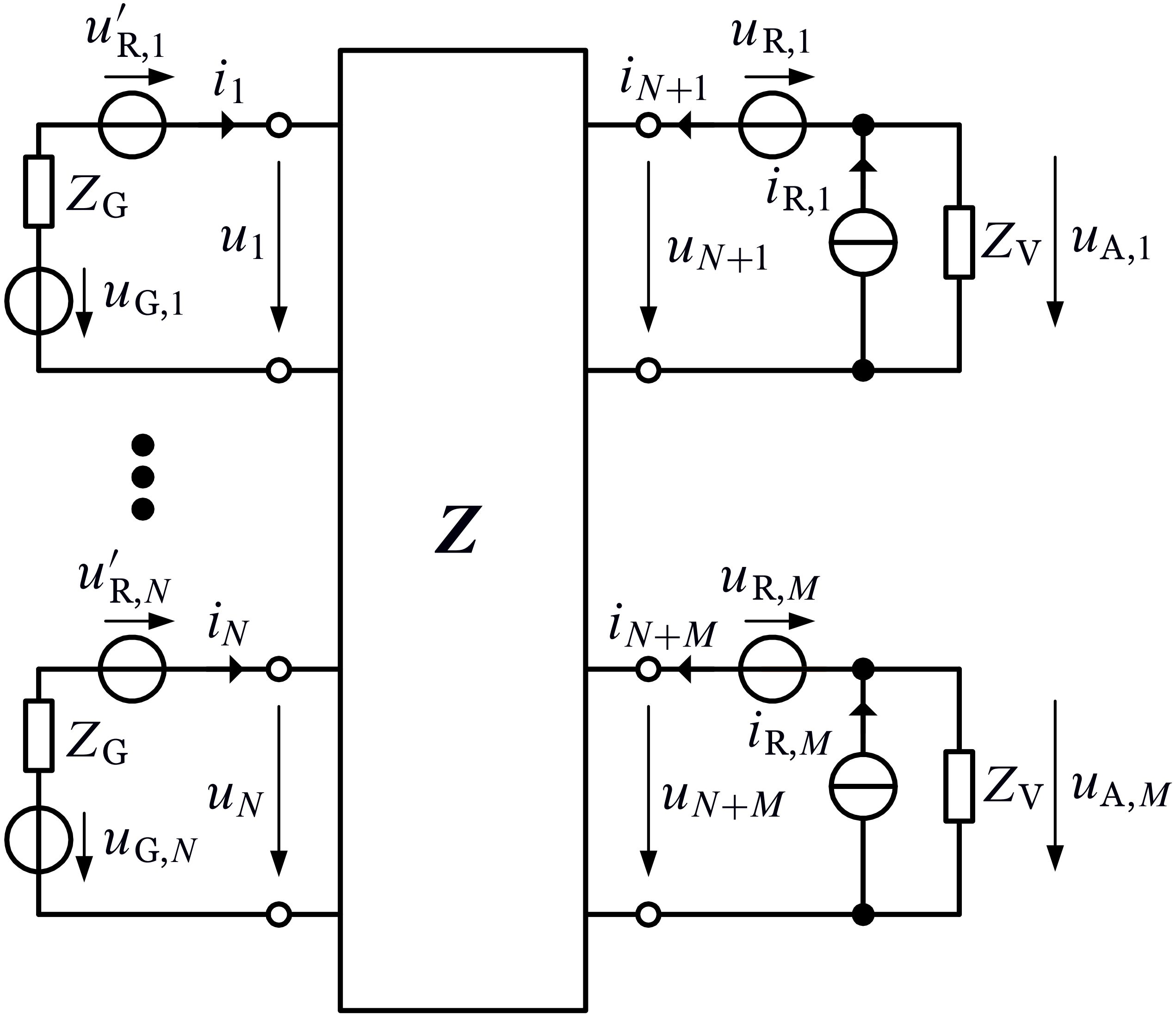
Modeling the effects of noise at the amplifier's input

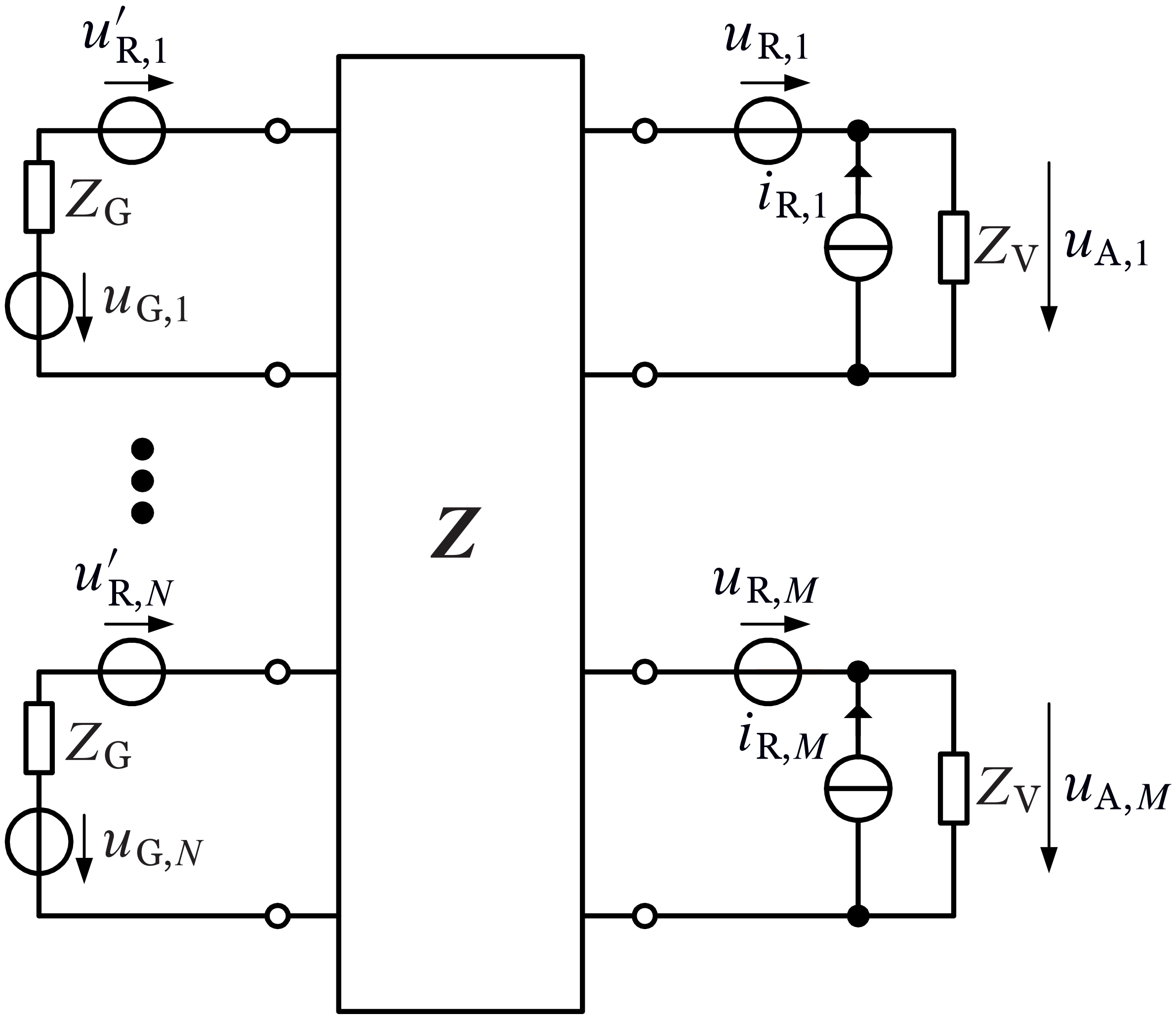


- Strictly linear $(N+M)$–port with 2 sources and 1 impedance per port



- Strictly linear $(N+M)$–port with 2 sources and 1 impedance per port

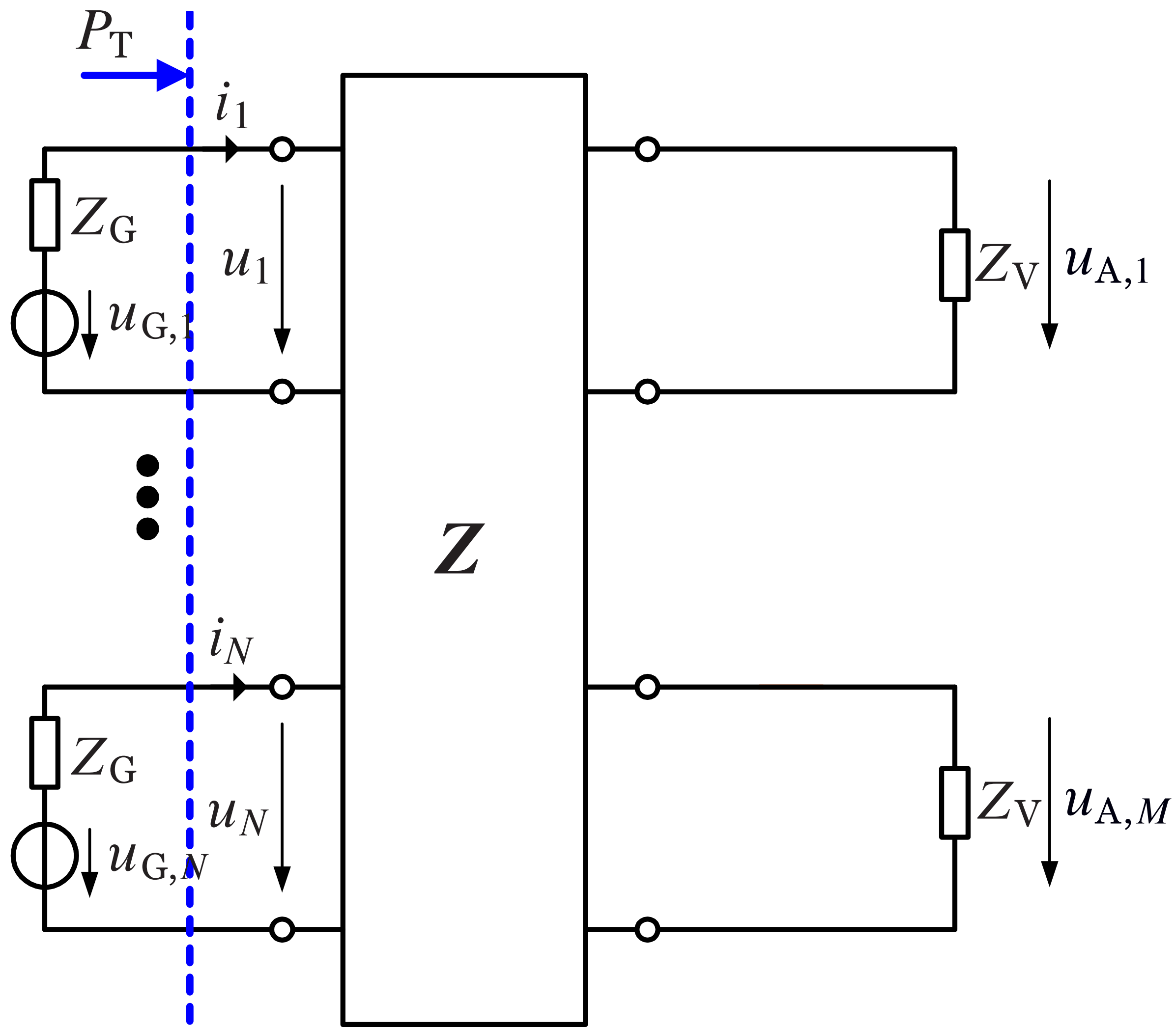




- Transfer properties:

$$\boldsymbol{u}_{\mathrm{A}} = \boldsymbol{D}\boldsymbol{u}_{\mathrm{G}} + \boldsymbol{\eta}$$

$$\boldsymbol{D} = \boldsymbol{D}\left(\boldsymbol{Z}, Z_{\mathrm{G}}, Z_{\mathrm{V}}\right) \in \mathbb{C}^{M \times N}$$



- Transfer properties:

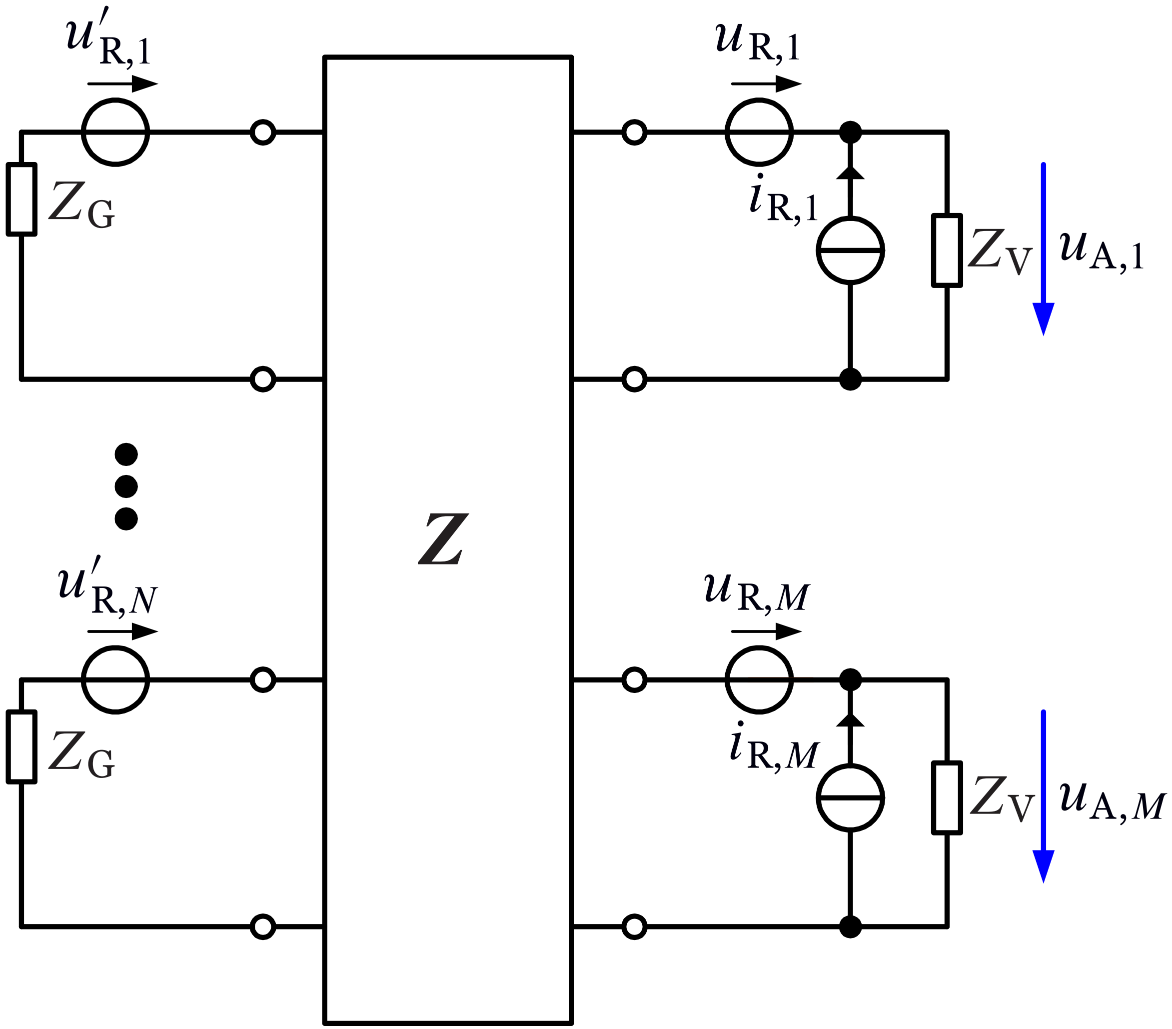
$$\boldsymbol{u}_\mathrm{A} = \boldsymbol{D}\boldsymbol{u}_\mathrm{G} + \boldsymbol{\eta}$$

$$\boldsymbol{D} = \boldsymbol{D}\left(\boldsymbol{Z}, Z_\mathrm{G}, Z_\mathrm{V}\right) \in \mathbb{C}^{M\times N}$$

- Transmit power:

$$P_\mathrm{T} = \boldsymbol{u}_\mathrm{G}^\mathrm{H}\boldsymbol{B}\boldsymbol{u}_\mathrm{G}$$

$$\boldsymbol{B} = \boldsymbol{B}\left(\boldsymbol{Z}, Z_\mathrm{G}, Z_\mathrm{V}\right) \in \mathbb{C}^{N\times N} \cdot \frac{\mathrm{A}}{\mathrm{V}}$$



- **Transfer properties:**

$$\boldsymbol{u}_A = \boldsymbol{D}\boldsymbol{u}_G + \boldsymbol{\eta}$$

$$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{Z}, Z_G, Z_V) \in \mathbb{C}^{M \times N}$$

- **Transmit power:**

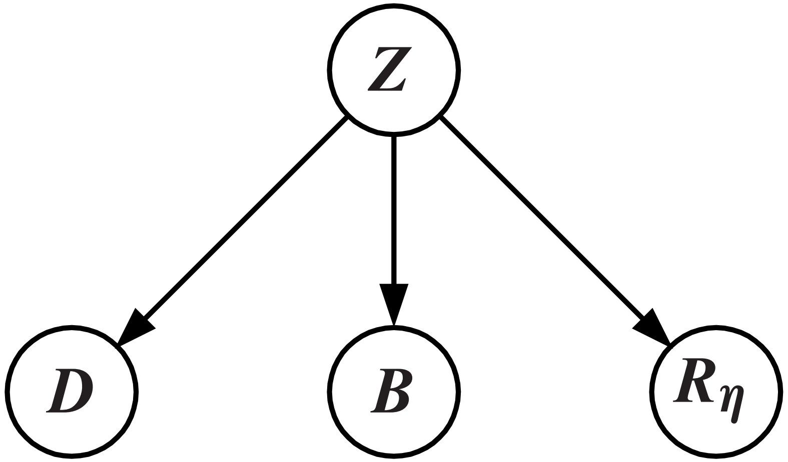
$$P_T = \boldsymbol{u}_G^H \boldsymbol{B} \boldsymbol{u}_G$$

$$\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{Z}, Z_G, Z_V) \in \mathbb{C}^{N \times N} \cdot \frac{A}{V}$$

- **Noise covariance:**

$$\boldsymbol{R}_\eta = \mathrm{E}\left[\boldsymbol{u}_A \boldsymbol{u}_A^H \mid \boldsymbol{u}_G = \boldsymbol{0}\right]$$

$$\boldsymbol{R}_\eta = \boldsymbol{R}_\eta(\boldsymbol{Z}, Z_G, Z_V, \text{noise par.}) \in \mathbb{C}^{M \times M} \cdot V^2$$



When we change even a single physical parameter of the system, we usually observe a simultaneous change in each and every of the 3 system matrices ($\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{R}_\eta$).

- Transfer properties:

$$\boldsymbol{u}_\mathrm{A} = \boldsymbol{D}\boldsymbol{u}_\mathrm{G} + \boldsymbol{\eta}$$

$$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{Z}, Z_\mathrm{G}, Z_\mathrm{V}) \in \mathbb{C}^{M\times N}$$

- Transmit power:

$$P_\mathrm{T} = \boldsymbol{u}_\mathrm{G}^\mathrm{H}\boldsymbol{B}\boldsymbol{u}_\mathrm{G}$$

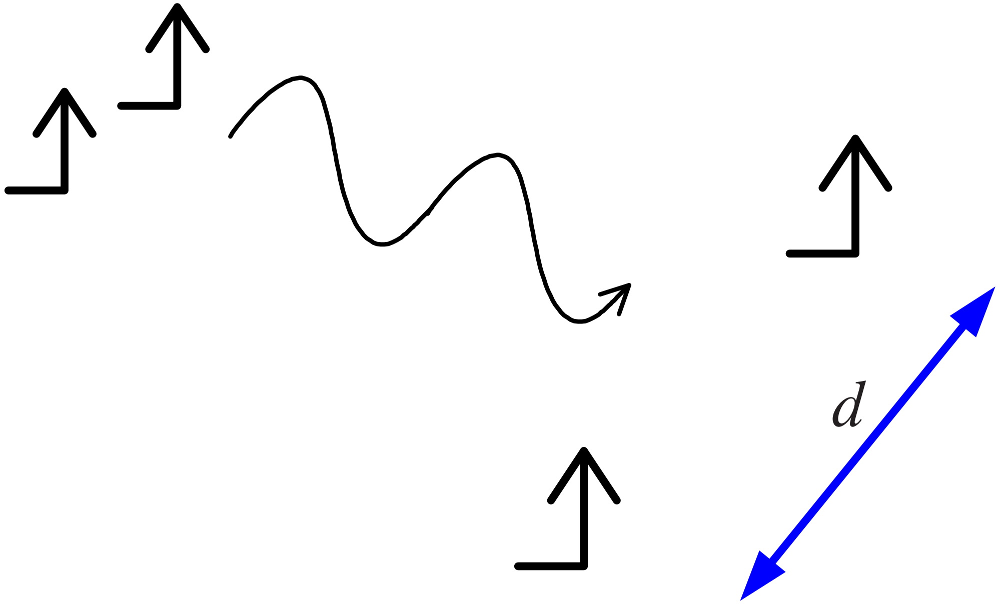
$$\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{Z}, Z_\mathrm{G}, Z_\mathrm{V}) \in \mathbb{C}^{N\times N} \cdot \frac{\mathrm{A}}{\mathrm{V}}$$

- Noise covariance

$$\boldsymbol{R}_\eta = \mathrm{E}[\boldsymbol{u}_\mathrm{A}\boldsymbol{u}_\mathrm{A}^\mathrm{H} \mid \boldsymbol{u}_\mathrm{G} = \boldsymbol{0}]$$

$$\boldsymbol{R}_\eta = \boldsymbol{R}_\eta(\boldsymbol{Z}, Z_\mathrm{G}, Z_\mathrm{V}, \text{noise par.}) \in \mathbb{C}^{M\times M} \cdot \mathrm{V}^2$$

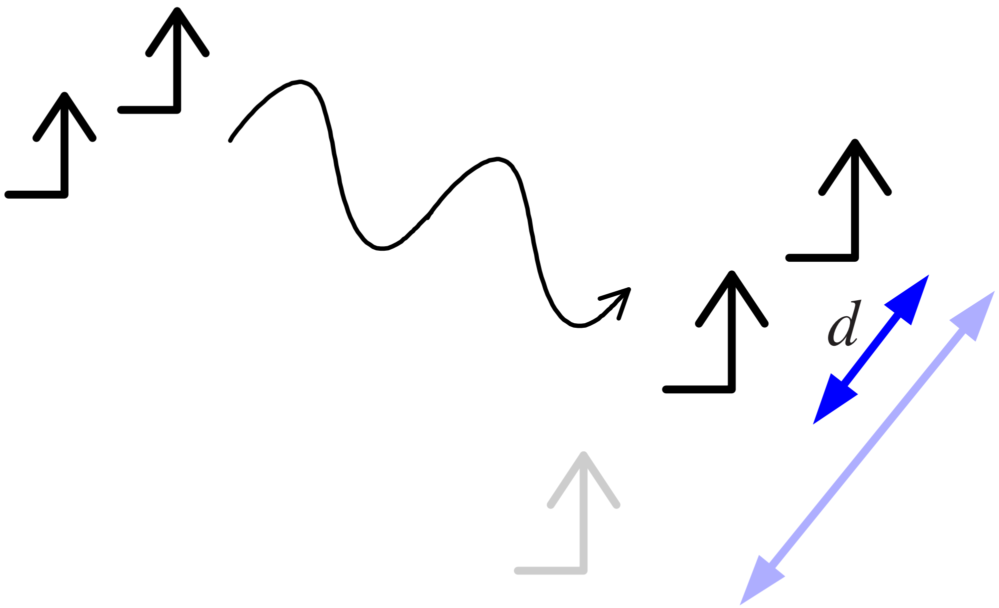
Example



$$\boldsymbol{u}_\mathrm{A} = \boldsymbol{D}\boldsymbol{u}_\mathrm{G} + \boldsymbol{\eta}$$

$$P_\mathrm{T} = \boldsymbol{u}_\mathrm{G}^\mathrm{H}\boldsymbol{B}\boldsymbol{u}_\mathrm{G}$$

$$\boldsymbol{\eta} \sim \mathcal{N}_\mathbb{C}\left(\boldsymbol{0}, \boldsymbol{R}_{\boldsymbol{\eta}}\right)$$



$$u_A = D u_G + \eta$$

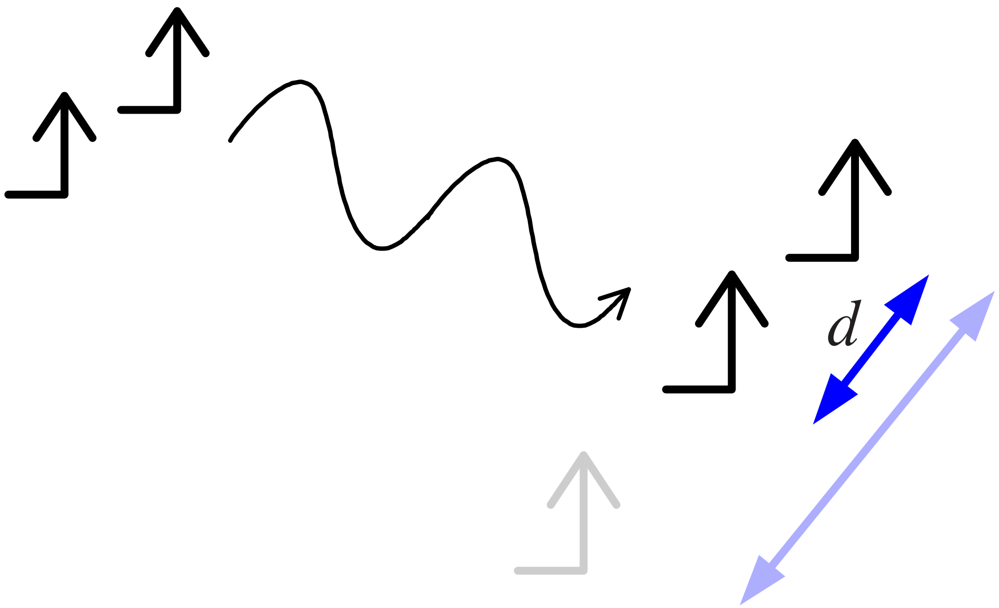
$$P_T = u_G^H B u_G$$

$$\eta \sim \mathcal{N}_{\mathbb{C}}(0, R_\eta)$$

* Change in one system parameter (d):

Simultaneous change in

1. transfer properties: D
2. noise covariance: R_η
3. transmit power coupling: B



$$\boldsymbol{u}_{\mathrm{A}} = \boldsymbol{D}\boldsymbol{u}_{\mathrm{G}} + \boldsymbol{\eta}$$

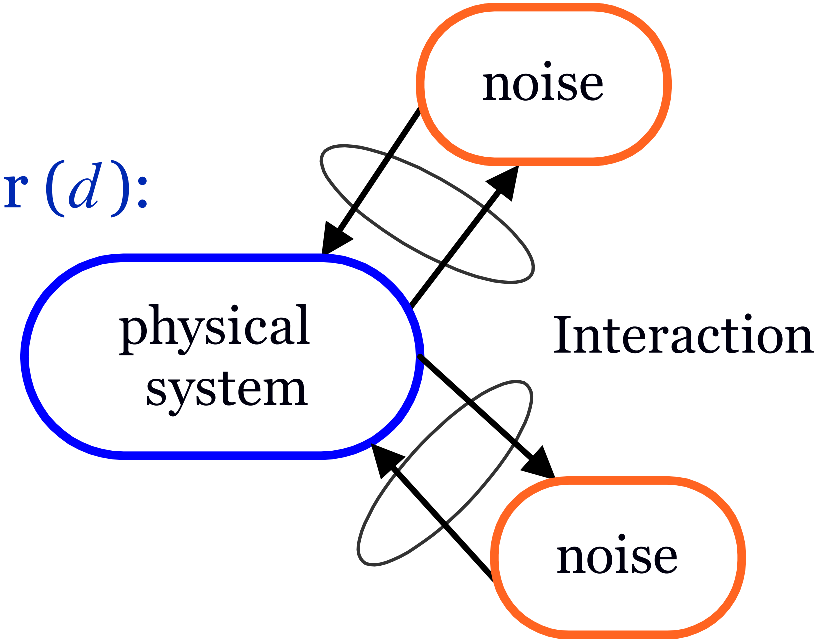
$$P_{\mathrm{T}} = \boldsymbol{u}_{\mathrm{G}}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{u}_{\mathrm{G}}$$

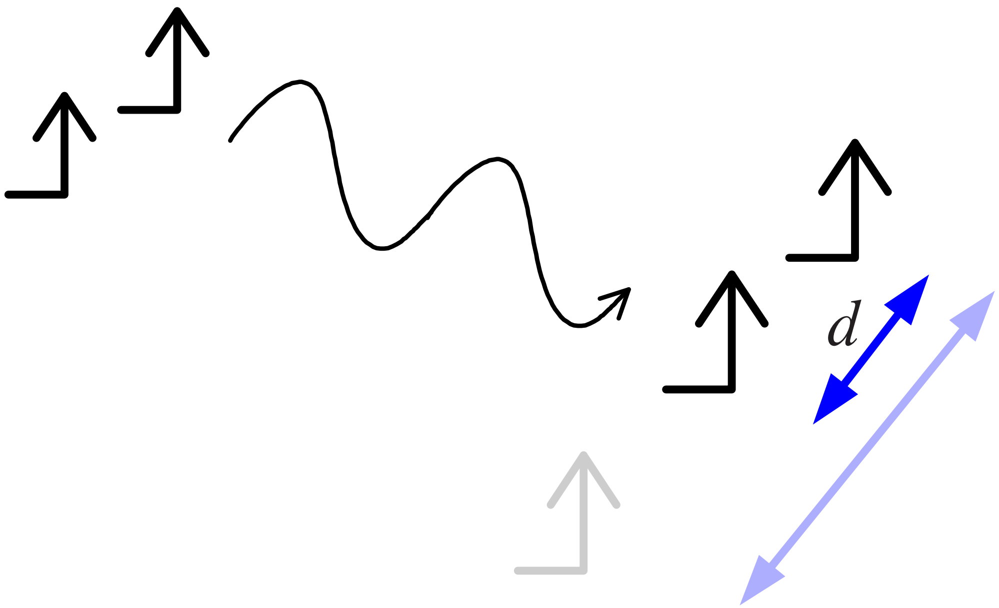
$$\boldsymbol{\eta} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \boldsymbol{R}_{\boldsymbol{\eta}}\right)$$

* Change in one system parameter (d):

  Simultaneous change in

  1. transfer properties: $\boldsymbol{D}$
  2. noise covariance: $\boldsymbol{R}_{\boldsymbol{\eta}}$
  3. transmit power coupling: $\boldsymbol{B}$

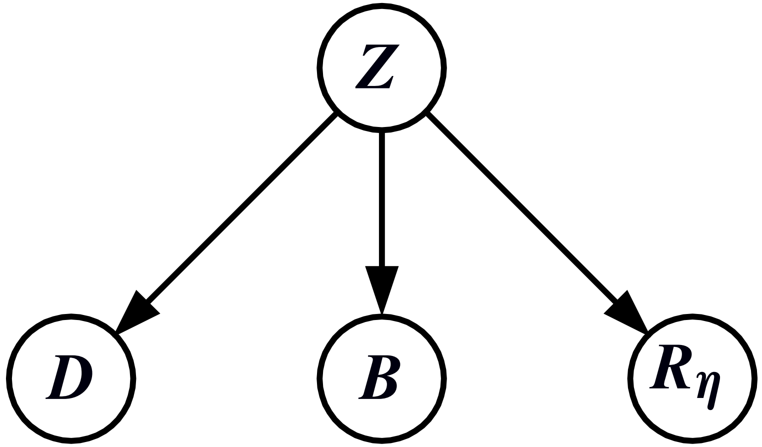




$$u_A = D u_G + \eta$$

$$P_T = u_G^H B u_G$$

$$\eta \sim \mathcal{N}_{\mathbb{C}}(0, R_\eta)$$



* Change in one system parameter (d):

  Simultaneous change in

  1. transfer properties: D
  2. noise covariance: R_η
  3. transmit power coupling: B

- Signal model:

$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\vartheta} \\
\boldsymbol{\vartheta} &\sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \sigma_{\vartheta}^{2}\mathbf{I}_{M}\right) \\
& \quad \sigma_{\vartheta}^{2} \text{ independent of } \boldsymbol{H} \\
P_{\mathrm{T}} &= ||\boldsymbol{x}||_{2}^{2}
\end{aligned}
$$

Signal assignment and channel matrix

- Signal model:

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\vartheta}$$
$$\boldsymbol{\vartheta} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \sigma_{\vartheta}^2 \mathbf{I}_M\right)$$
$$\sigma_{\vartheta}^2 \text{ independent of } \boldsymbol{H}$$
$$P_{\mathrm{T}} = ||\boldsymbol{x}||_2^2$$

- Multiport model:

$$\boldsymbol{u}_{\mathrm{A}} = \boldsymbol{D}\boldsymbol{u}_{\mathrm{G}} + \boldsymbol{\eta}$$
$$\boldsymbol{\eta} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \boldsymbol{R}_{\boldsymbol{\eta}}\right)$$
$$P_{\mathrm{T}} = \boldsymbol{u}_{\mathrm{G}}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{u}_{\mathrm{G}}$$

Signal assignment and channel matrix

- **Signal model:**

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\vartheta}$$

$$\boldsymbol{\vartheta} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \sigma_{\vartheta}^2 \mathbf{I}_M\right)$$

σ_{ϑ}^2 independent of $\boldsymbol{H}$

$$P_{\mathrm{T}} = ||\boldsymbol{x}||_2^2$$

- **Multiport model:**

$$\boldsymbol{u}_{\mathrm{A}} = \boldsymbol{D}\boldsymbol{u}_{\mathrm{G}} + \boldsymbol{\eta}$$

$$\boldsymbol{\eta} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \boldsymbol{R}_{\boldsymbol{\eta}}\right)$$

$$P_{\mathrm{T}} = \boldsymbol{u}_{\mathrm{G}}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{u}_{\mathrm{G}}$$

- **Signal assignment:**

$$\boldsymbol{x} = \boldsymbol{V}\boldsymbol{u}_{\mathrm{G}}$$

$$\boldsymbol{y} = \boldsymbol{W}^{-1}\boldsymbol{u}_{\mathrm{L}}$$

Signal assignment and channel matrix

Signal model:

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\vartheta}$$

$$\boldsymbol{\vartheta} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \sigma_{\vartheta}^2 \mathbf{I}_M\right)$$

$$\sigma_{\vartheta}^2 \text{ independent of } \boldsymbol{H}$$

$$P_{\mathrm{T}} = ||\boldsymbol{x}||_2^2$$

Signal assignment:

$$\boldsymbol{x} = \boldsymbol{V}\boldsymbol{u}_{\mathrm{G}}$$

$$\boldsymbol{y} = \boldsymbol{W}^{-1}\boldsymbol{u}_{\mathrm{L}}$$

$$\boldsymbol{H} = \boldsymbol{W}^{-1}\boldsymbol{D}\boldsymbol{V}^{-1}$$

Multiport model:

$$\boldsymbol{u}_{\mathrm{A}} = \boldsymbol{D}\boldsymbol{u}_{\mathrm{G}} + \boldsymbol{\eta}$$

$$\boldsymbol{\eta} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \boldsymbol{R}_{\boldsymbol{\eta}}\right)$$

$$P_{\mathrm{T}} = \boldsymbol{u}_{\mathrm{G}}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{u}_{\mathrm{G}}$$

Signal assignment and channel matrix

* Signal model:

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\vartheta}$$
$$\boldsymbol{\vartheta} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \sigma_\vartheta^2 \mathbf{I}_M\right)$$
$$\sigma_\vartheta^2 \text{ independent of } \boldsymbol{H}$$
$$P_{\mathrm{T}} = ||\boldsymbol{x}||_2^2$$

* Signal assignment:

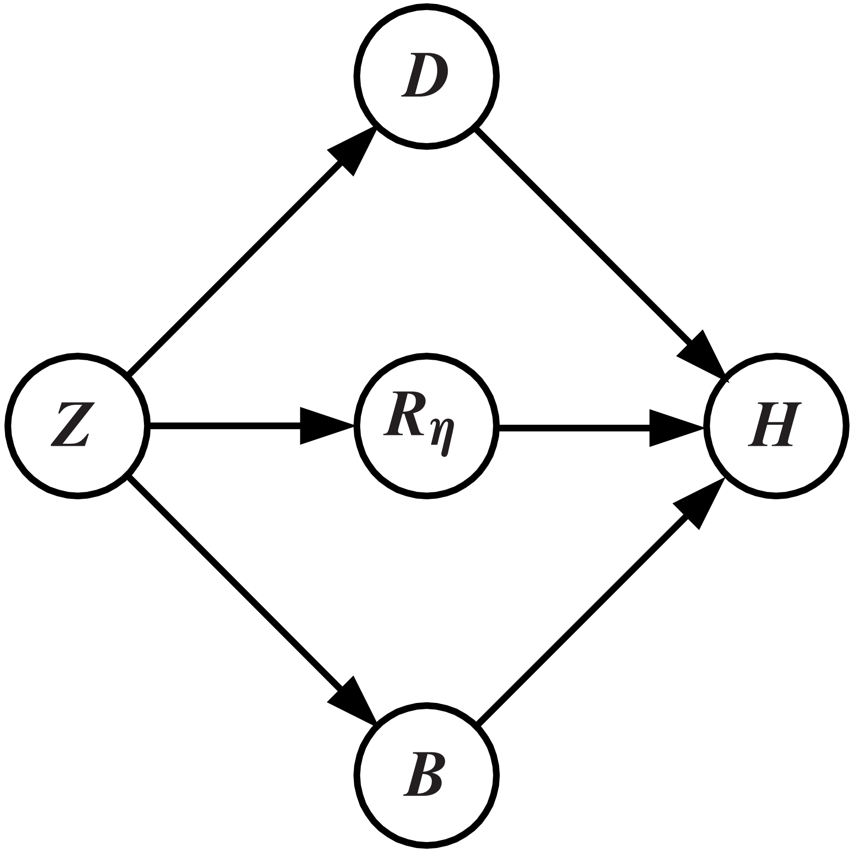
$$\boldsymbol{x} = \boldsymbol{V}\boldsymbol{u}_{\mathrm{G}}$$
$$\boldsymbol{y} = \boldsymbol{W}^{-1}\boldsymbol{u}_{\mathrm{L}}$$
$$\boldsymbol{H} = \boldsymbol{W}^{-1}\boldsymbol{D}\boldsymbol{V}^{-1}$$

* Multiport model:

$$\boldsymbol{u}_{\mathrm{A}} = \boldsymbol{D}\boldsymbol{u}_{\mathrm{G}} + \boldsymbol{\eta}$$
$$\boldsymbol{\eta} \sim \mathcal{N}_{\mathbb{C}}\left(\mathbf{0}, \boldsymbol{R}_{\boldsymbol{\eta}}\right)$$
$$P_{\mathrm{T}} = \boldsymbol{u}_{\mathrm{G}}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{u}_{\mathrm{G}}$$

* Equivalence of both models:

$$\boldsymbol{W} = \boldsymbol{R}_{\boldsymbol{\eta}}^{1/2} / \sqrt{\sigma_\vartheta^2}$$
$$\boldsymbol{V} = \boldsymbol{B}^{1/2}, \quad \boldsymbol{B} > \mathbf{0}$$
$$\boldsymbol{H} = \sqrt{\sigma_\vartheta^2}\,\boldsymbol{R}_{\boldsymbol{\eta}}^{-1/2}\boldsymbol{D}\boldsymbol{B}^{-1/2}$$



$$\boldsymbol{H} = \sqrt{\sigma_{\vartheta}^{2}}\, \boldsymbol{R}_{\boldsymbol{\eta}}^{-1/2} \boldsymbol{D}\, \boldsymbol{B}^{-1/2}$$

contains all the relevant physics about

1. signal transfer
2. effective noise
3. transmit power

Application: compact antenna arrays

- Two linear antenna arrays of 4 isotropic radiators each
- Antenna separation inside arrays: d

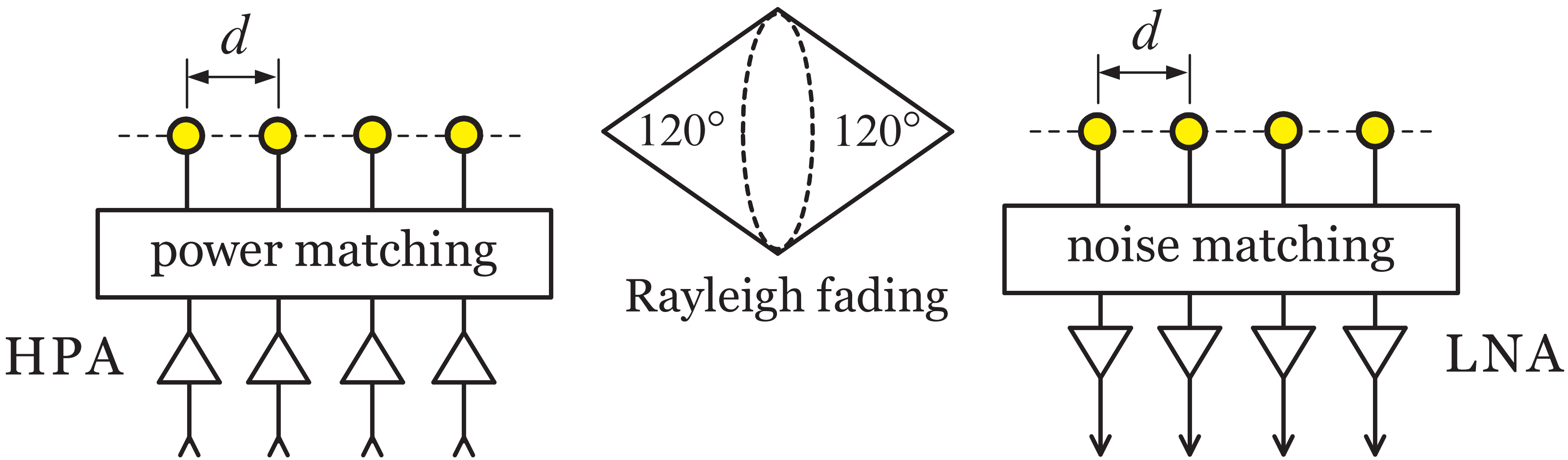
Application: compact antenna arrays

- Two linear antenna arrays of 4 isotropic radiators each
- Antenna separation inside arrays: d

- Two linear antenna arrays of 4 isotropic radiators each
- Antenna separation inside arrays: d
- Impedance matching networks at transmitter and receiver

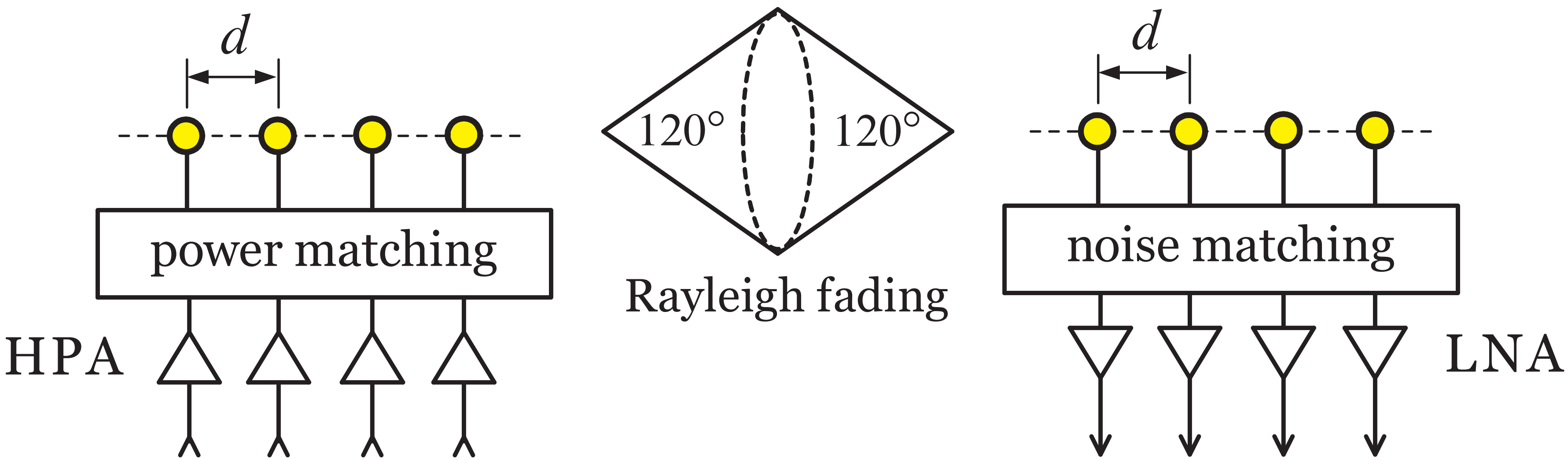
Application: compact antenna arrays

- Two linear antenna arrays of 4 isotropic radiators each
- Antenna separation inside arrays: d
- Impedance matching networks at transmitter and receiver



Application: compact antenna arrays

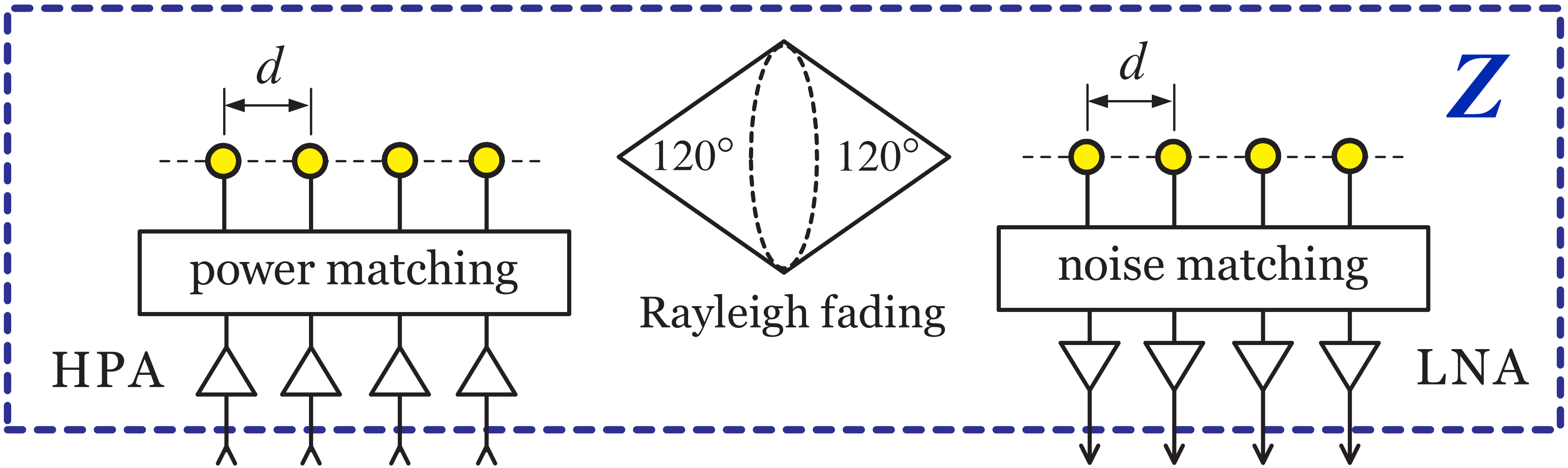
- Two linear antenna arrays of 4 isotropic radiators each
- Antenna separation inside arrays: d
- Impedance matching networks at transmitter and receiver



✻ Question: what happens to the rank of $\boldsymbol{H}$ in case that $d \to 0$?

Application: compact antenna arrays

- Two linear antenna arrays of 4 isotropic radiators each
- Antenna separation inside arrays: d
- Impedance matching networks at transmitter and receiver



✻ Question: what happens to the rank of $\boldsymbol{H}$ in case that $d \to 0$?

* Singular values of $\boldsymbol{H}$:

$$s_1 \geq s_2 \geq s_3 \geq \cdots$$

- Singular values of $\boldsymbol{H}$:

$$s_1 \geq s_2 \geq s_3 \geq \cdots$$

- Define:

$$\Psi = \mathrm{E}\left[\frac{s_2}{s_1}\right]$$

- Singular values of $\boldsymbol{H}$:

$$s_1 \geq s_2 \geq s_3 \geq \cdots$$

- Define:

$$\Psi = \mathrm{E}\left[\frac{s_2}{s_1}\right]$$

- Question:

How does Ψ depend in d ?

- **Singular values of $\boldsymbol{H}$:**

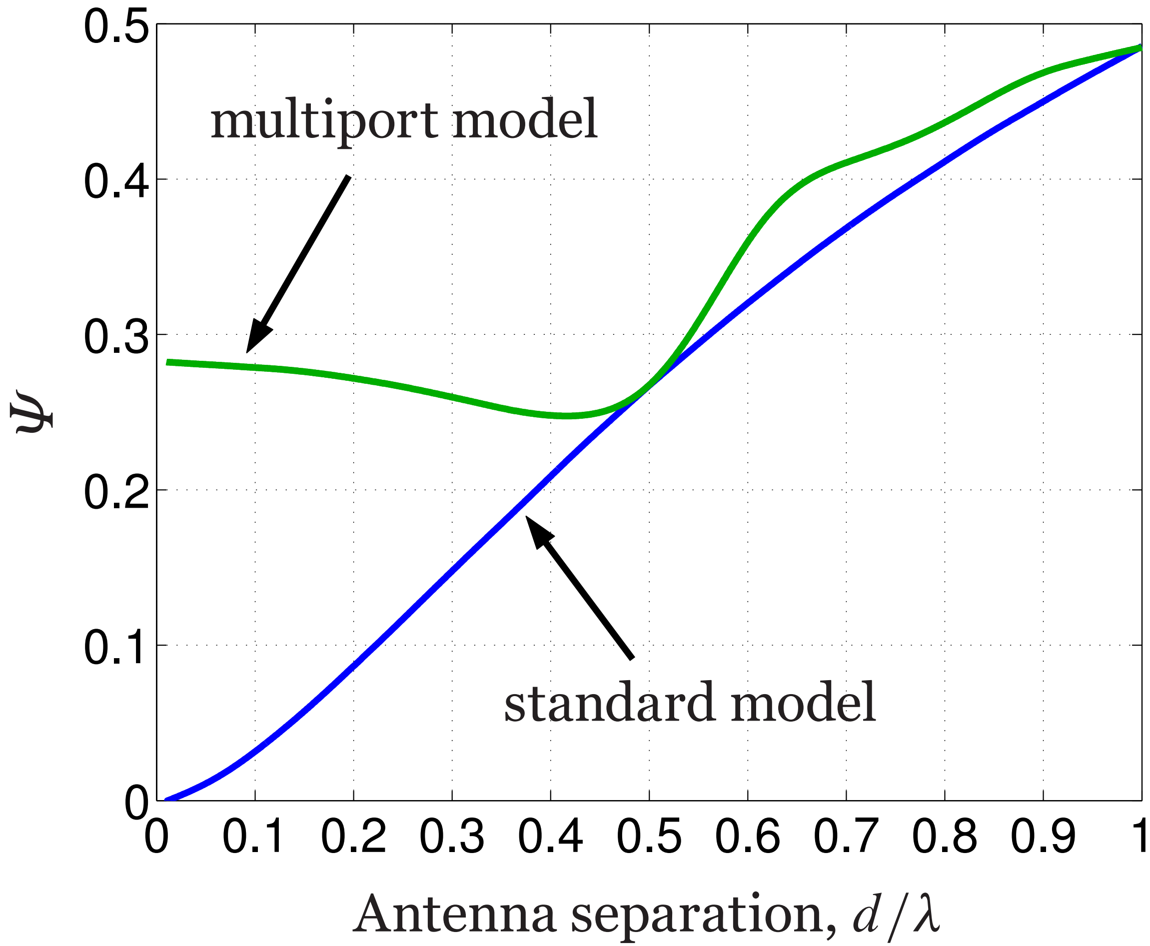
$$s_1 \geq s_2 \geq s_3 \geq \cdots$$

- **Define:**

$$\Psi = \mathrm{E}\left[\frac{s_2}{s_1}\right]$$

- **Question:**

How does Ψ depend on d ?



- Singular values of $\boldsymbol{H}$:

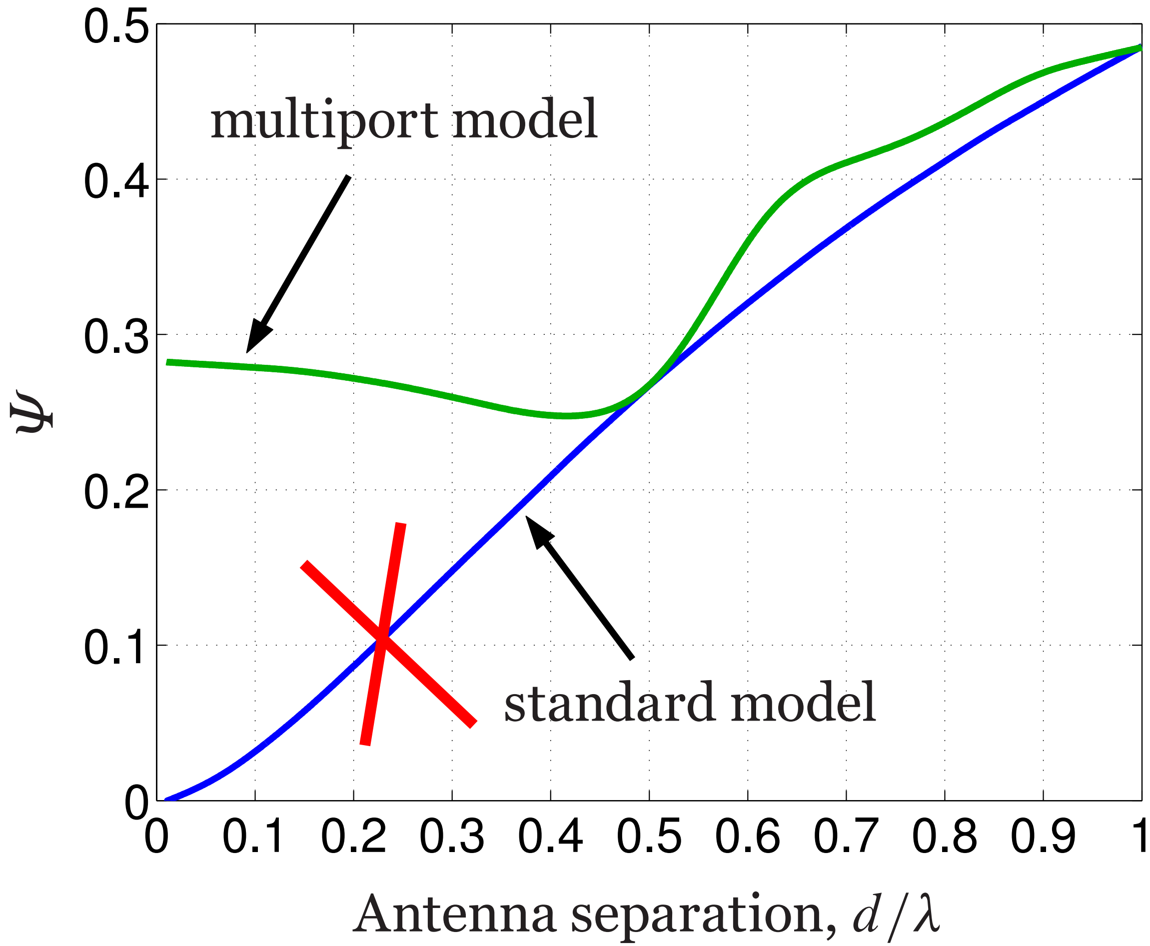
$$s_1 \geq s_2 \geq s_3 \geq \cdots$$

- Define:

$$\Psi = \mathrm{E}\left[\frac{s_2}{s_1}\right]$$

- Question:

How does Ψ depend on d ?



- Parallel data streams possible even with compact antenna arrays!

Rudy Kalman (2005):*

*Plenary lecture, International Federation of Automatic Control World Congress, Prague, 04. July 2005

Rudy Kalman (2005):*

1. Get the physics right.
2. The rest is mathematics.

* Plenary lecture, International Federation of Automatic Control World Congress, Prague, 04. July 2005

Rudy Kalman (2005):*

1. Get the physics right.
2. The rest is mathematics.

Thank you!

*Plenary lecture, International Federation of Automatic Control World Congress, Prague, 04. July 2005