On the Relationship between Fully and Locally Connected Oscillatory Networks

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Introduction and Motivation

- Oscillatory networks present neurocomputational properties similar to those of Hopfield networks, with the only difference that stored patterns are not equilibria, but synchronized oscillatory states with suitable phase relations;
- An approximated expression of the law governing the evolution of the phase (phase equation) of each oscillator can be obtained under the condition of weak coupling among the oscillators;
- Equilibria of the phase equation, described by a fully connected networks with non-monotonic activation function[a], are in one-to-one correspondence with global periodic oscillations of the oscillatory networks ⇒ associative memories based on limit cycles

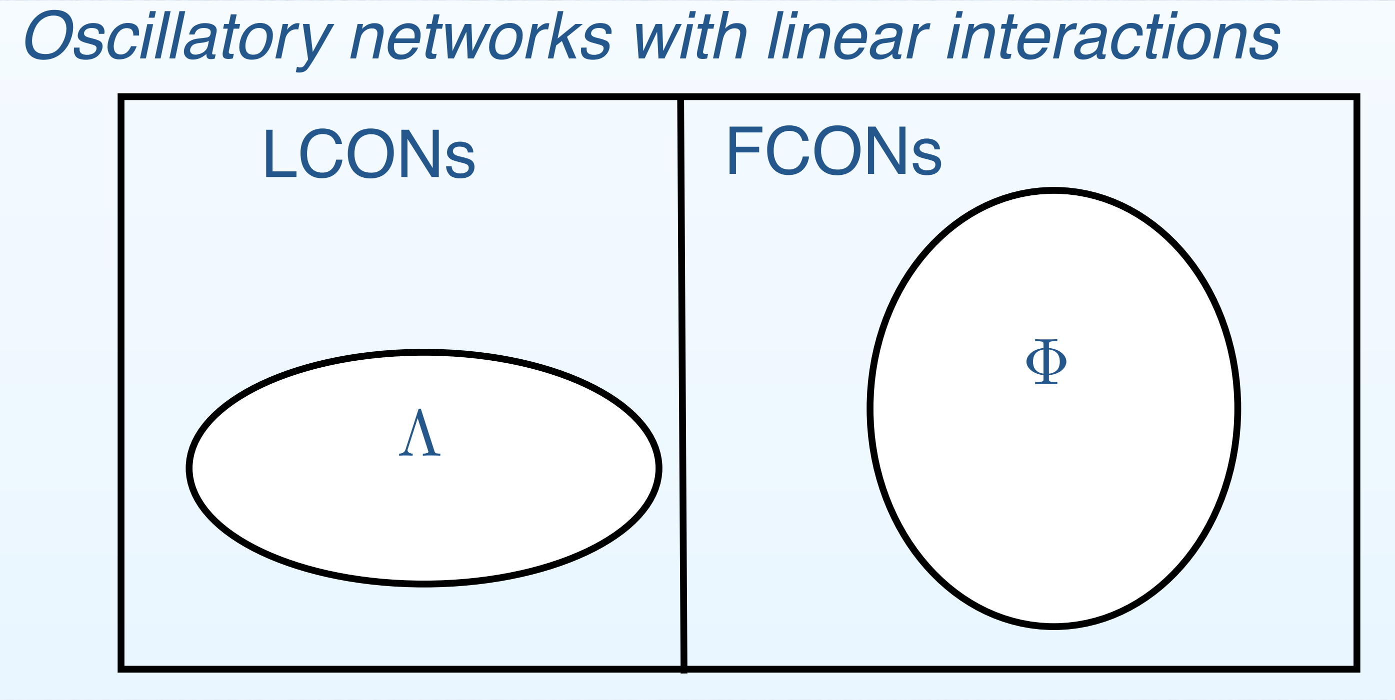
(a) non-monotonic activation functions ⇒ higher storage capacity

Oscillatory Networks

- The main bottleneck of Fully Connected Oscillatory Networks (FCONs) with N units is that N^2 connections are required.
- Locally Connected Oscillatory Networks (LCONs) are more suitable for integrated circuit implementations due to the properties of local connectivity and fixed "interaction weights".
- trade-off between the number of connections and functionality properties.

Aim of the work

- Are there conditions such that a given LCON $\in \mathbf{\Lambda}$ can be regarded as a FCON $\in \mathbf{\Phi}$? $\Rightarrow$ Intuitively, LCONs ~ FCONs.

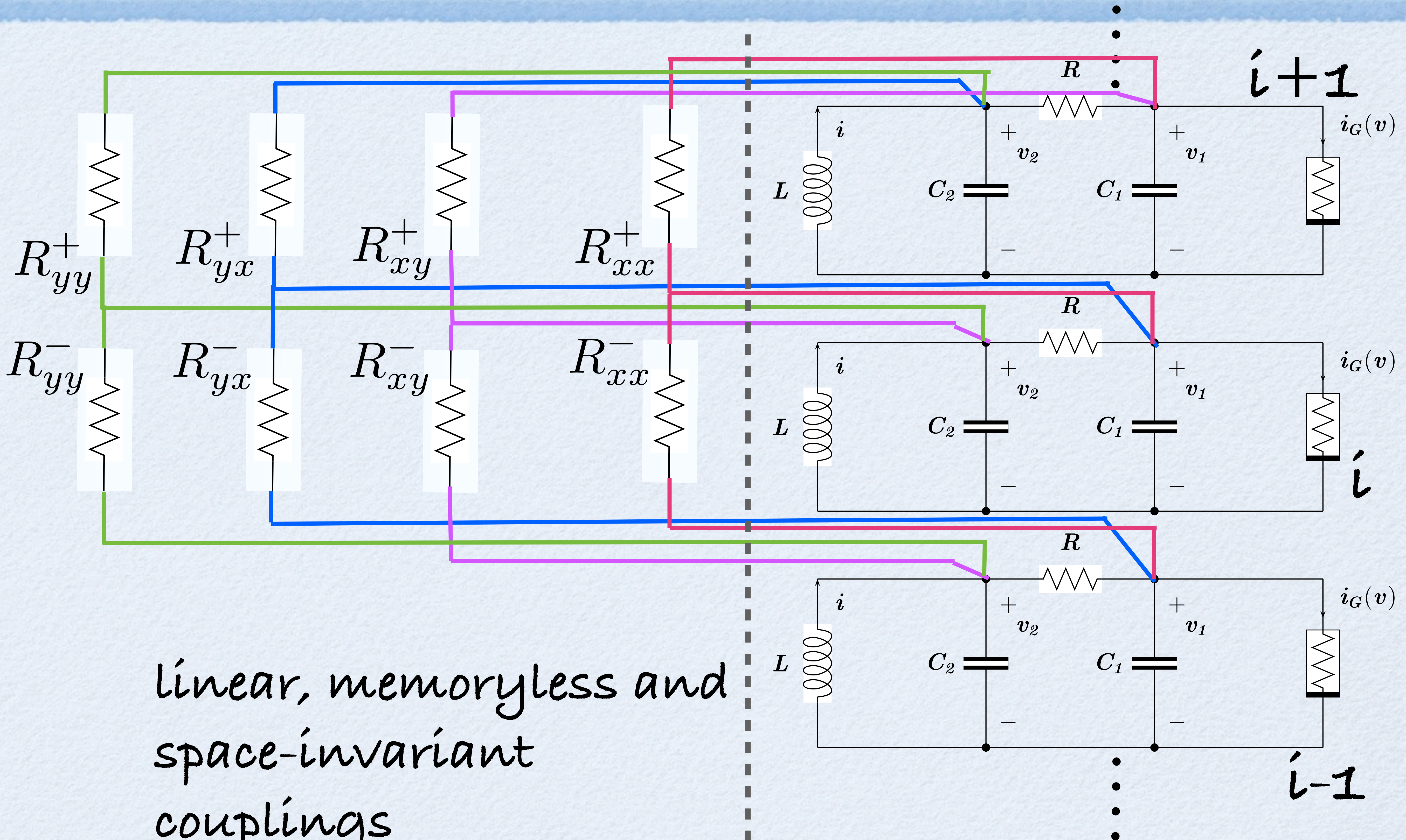


- $\mathbf{\Lambda}$ = {set of locally connected oscillatory networks (LCONs) with linear, memoryless and space-invariant couplings};
- $\mathbf{\Phi}$ = {set of fully connected oscillatory networks (FCONs) with linear dynamic interactions}

One-dimensional networks

- (A1) each cell can be suitably separated into linear and nonlinear blocks (Lur'e model);
- (A2) there exists at least a global periodic oscillation with angular frequency ω;
- (A3) interactions are linear.
    - LCONs: the i^{th} cell is coupled to its nearest neighbours ($N_i = \{i-1, i+1\}$) by using only two component of the state vector, i.e. $X_i = [x_i, y_i, \mathbf{z}_i] \in \mathbf{R}^m \ (i = 1, \ldots, N)$
    - FCONs: the ith cell is coupled to all the others, i.e. interactions may be described by $Y_{ik}(D)x_i, \ \forall k = 1, \ldots, N$
    - time-differential operator $D = \dfrac{d}{dt}$

1-D array of N identical Chua's circuits



linear, memoryless and space-invariant couplings

LCONs

$$\dot{X}_i = F(X_i) + G(X_k, X_i) \quad (1 \leq i \leq N,\, k = i \pm 1)$$

$$X_i = [x_i, y_i, \mathbf{z}_i]'(\mathbf{z}_i \in \mathbf{R}^{m-2})$$

$$G(X_k, X_i) = [g_x, g_y, 0, \ldots, 0]' \in \mathbf{R}^m$$

$$\begin{aligned} g_x &= C^+_{xx}(x_{i+1} - x_i) + C^+_{yx}(y_{i+1} - x_i) \\ &\quad + C^-_{xx}(x_{i-1} - x_i) + C^-_{yx}(y_{i-1} - x_i) \\ g_y &= C^+_{xy}(x_{i+1} - y_i) + C^+_{yy}(y_{i+1} - y_i) \\ &\quad + C^-_{xy}(x_{i-1} - y_i) + C^-_{yy}(y_{i-1} - y_i) \end{aligned}$$

- Coupling parameters $C^\pm_{xx}, C^\pm_{xy}, C^\pm_{yx}, C^\pm_{yy}$ are assumed to be space-invariant, linear and memoryless.

LCONs (conts)

- (A1) -> each cell $\dot{X}_i = F(X_i)$ admits of a Lur'e-like description:

$$\begin{aligned}
\dot{x}_i &= A_{11}x_i + A_{12}y_i + \mathbf{A}_{13}\mathbf{z}_i + f(x_i) + g_x \\
\dot{y}_i &= A_{21}x_i + A_{22}y_i + \mathbf{A}_{23}\mathbf{z}_i + g_y \\
\dot{\mathbf{z}}_i &= \mathbf{A}_{31}x_i + \mathbf{A}_{32}y_i + \mathbf{A}_{33}\mathbf{z}_i
\end{aligned}$$

$$A_{11}, A_{12}, A_{21}, A_{22} \in \mathbf{R}, \mathbf{A}_{13}, \mathbf{A}_{23} \in \mathbf{R}^{1,m-2},$$
$$\mathbf{A}_{31}, \mathbf{A}_{32} \in \mathbf{R}^{m-2,1}, \mathbf{A}_{33} \in \mathbf{R}^{m-2,m-2}$$

$f(\cdot)$ is the scalar Lipschitz nonlinear function

$$\mathbf{z}_i = [\mathbf{I}(D) - \mathbf{A}_{33}]^{-1}[\mathbf{A}_{31}x_i + \mathbf{A}_{32}y_i]$$

LCONs as FCONs

$$\mathbf{P}(D)\mathbf{x} = \mathbf{Q}(D)\mathbf{y} + f(\mathbf{x})$$

$$\mathbf{R}(D)\mathbf{y} = \mathbf{S}(D)\mathbf{x} \quad \Rightarrow \quad \mathbf{y} = \mathbf{R}^{-1}(D)\mathbf{S}(D)\mathbf{x}$$

$\mathbf{P}(D), \mathbf{Q}(D), \mathbf{R}(D), \mathbf{S}(D) \in \mathbf{R}^{N,N}$ are tridiagonal matrices

$$\begin{aligned}
\mathbf{P}(D) &= [(D+\sigma_x) - H_{11}(D)]\mathbf{I_N} - C_{xx}^{+}\mathbf{T} - C_{xx}^{-}\mathbf{T}' \\
\mathbf{Q}(D) &= H_{12}(D)\mathbf{I_N} + C_{yx}^{+}\mathbf{T} + C_{yx}^{-}\mathbf{T}' \\
\mathbf{R}(D) &= [(D+\sigma_y) - H_{22}(D)]\mathbf{I_N} - C_{yy}^{+}\mathbf{T} - C_{yy}^{-}\mathbf{T}' \\
\mathbf{S}(D) &= H_{21}(D)\mathbf{I_N} + C_{xy}^{+}\mathbf{T} + C_{xy}^{-}\mathbf{T}'
\end{aligned}$$

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\sigma_x = C_{xx}^{+} + C_{xx}^{-} + C_{yx}^{+} + C_{yx}^{-}, \; \sigma_y = C_{yy}^{+} + C_{yy}^{-} + C_{xy}^{+} + C_{xy}^{-}$$

$\mathbf{I_N} \in \mathbf{R}^{N,N}$ identity matrix

$$[L(D)\mathbf{I_N}]\mathbf{x} = f(\mathbf{x}) + [C_{xx}^{+}\mathbf{T} + C_{xx}^{-}\mathbf{T}' + \mathbf{Q}(D)\mathbf{R}^{-1}(D)\mathbf{S}(D)]\mathbf{x}$$

where $L(D) = [(D+\sigma_x) - H_{11}(D)]$

LCONs as FCONs

$$L(D)\mathbf{x} = f(\mathbf{x}) + \mathbf{Y}(D)\mathbf{x},\ \ \mathbf{Y}(D) = C_{xx}^{+}\mathbf{T} + C_{xx}^{-}\mathbf{T}' + \mathbf{Q}(D)\mathbf{R}^{-1}(D)\mathbf{S}(D)$$

$$L(D)x_i(t) = f[x_i(t)] + \sum_{k=1}^{N} Y_{ik}(D)x_k(t)$$

- Proposition 1: If $C_{yy}^{+}C_{yy}^{-} = 0$ then LCONs described by (1)-(3) exhibit no full connectivity properties.

- Proposition 2: If both C_{yy}^{+} and C_{yy}^{-} are different from zero then LCONs $\in \Lambda$, i.e. described by (1)-(3), acts as FCONs $\in \Phi$. Furthermore, if $C_{xx}^{\pm} = 0$, and $C_{xy}^{\pm} = 0$ then $C_{yx}^{\pm} = 0$ the explicit expression of the dynamic coupling between the i^{th} and j^{th} cells can be derived (see proceedings).

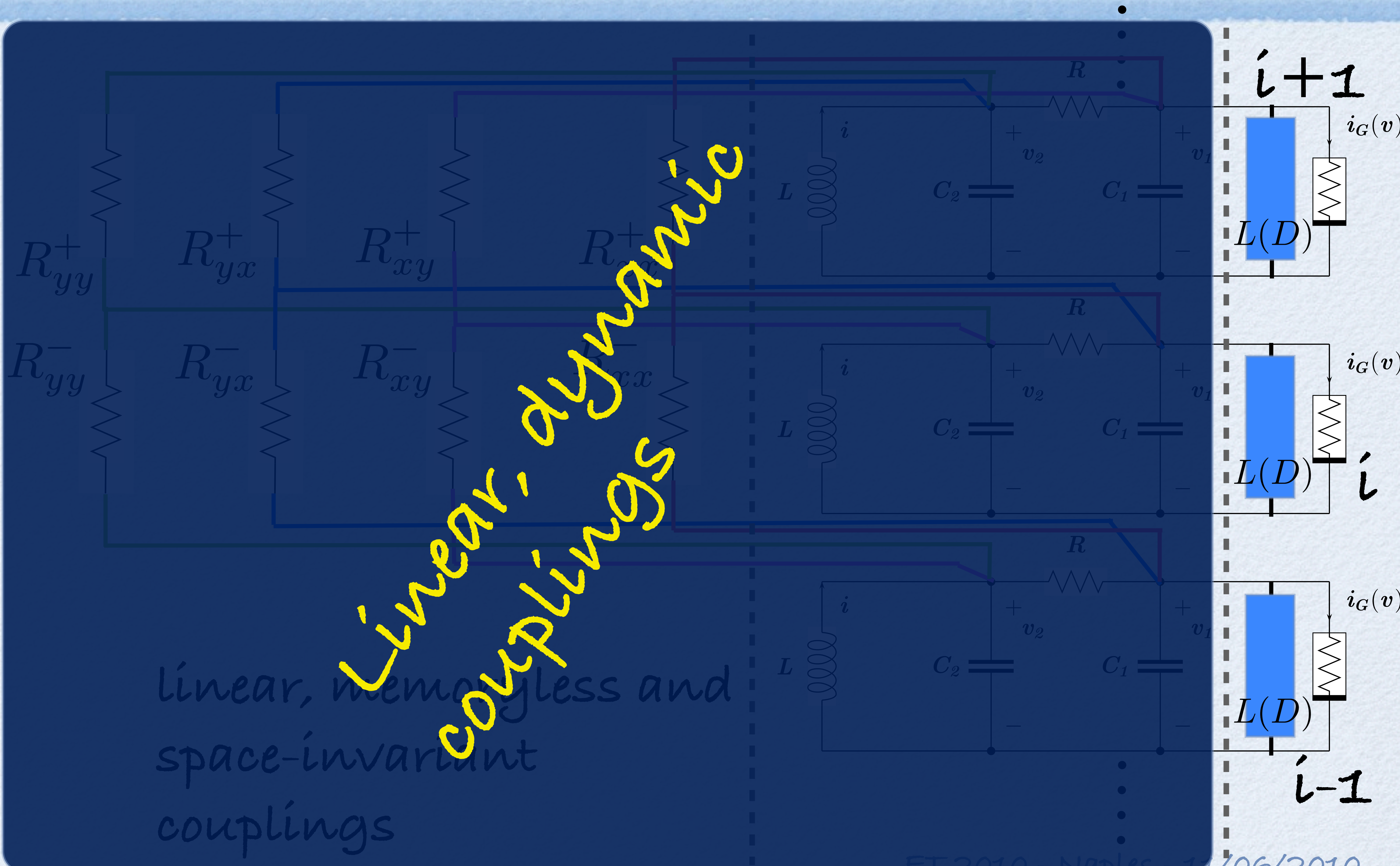
LCONs as FCONs - Remarks

- Equation $\mathbf{Y}(D) = C_{xx}^{+}\mathbf{T} + C_{xx}^{-}\mathbf{T}' + \mathbf{Q}(D)\mathbf{R}^{-1}(D)\mathbf{S}(D)$ makes clear full connectivity properties of LCONs

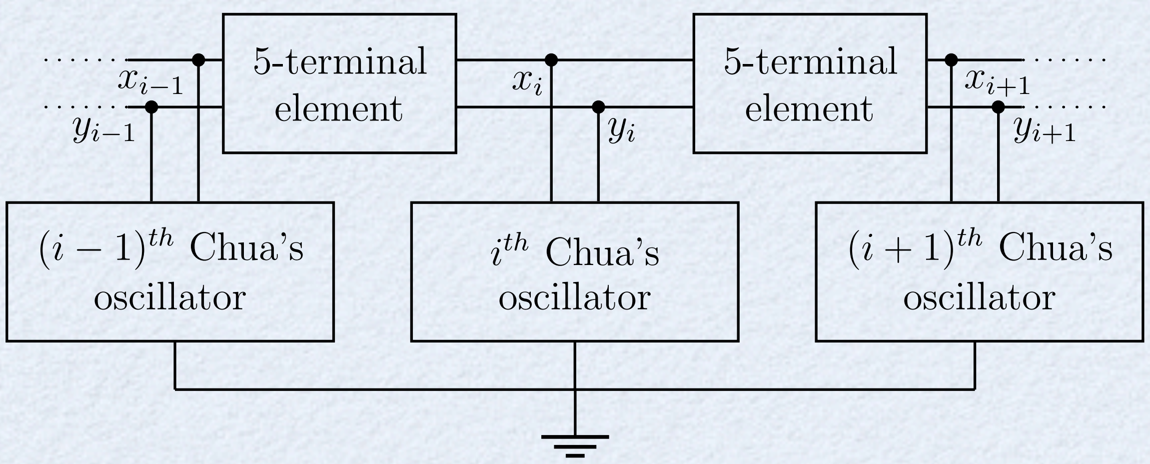
- $\mathbf{R}(D)$ depends mainly on uncoupled cell parameters and coupling coefficients $C_{yy}^{\pm}$ related to y_i. Its analytical evaluation allows us to establish a procedure for deriving a FCON from a given LCON.

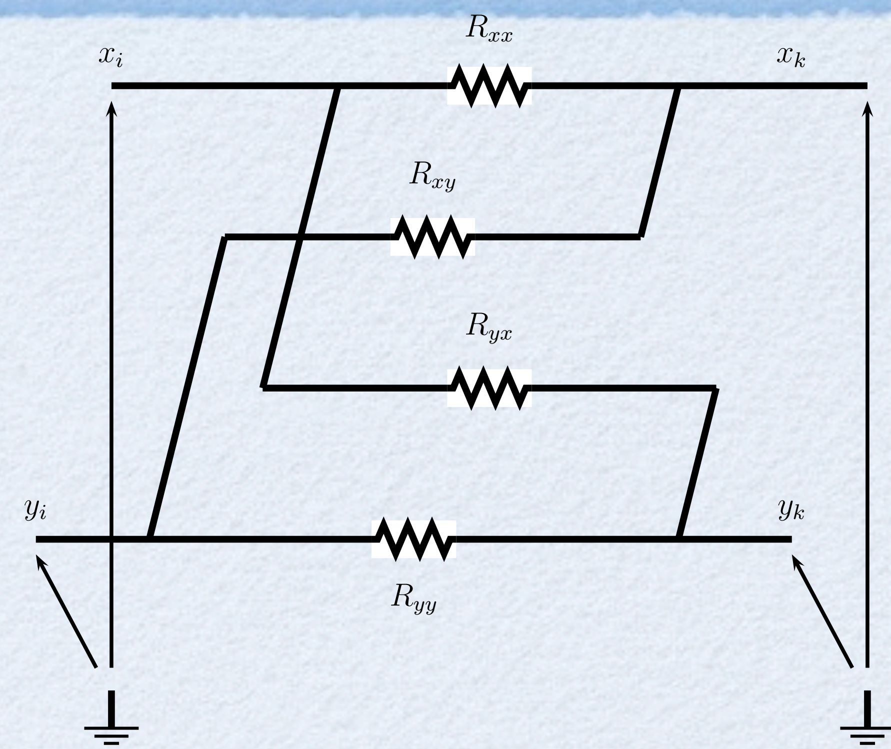
- $\mathbf{Y}(D) = C_{xx}^{+}\mathbf{T} + C_{xx}^{-}\mathbf{T}' + \mathbf{Q}(D)\mathbf{R}^{-1}(D)\mathbf{S}(D)$ provides some suggestions about the inverse procedure, i.e. how to draw a LCON realizing a given FCON.

1-D array of N identical Chua's circuits



1-D array of 4 identical Chua's circuits





$$X_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$$

$$F_i(X_i) = \begin{pmatrix} \alpha\,[y_i - x_i - f(x_i)] \\ x_i - y_i + z_i \\ -\beta\, y_i \end{pmatrix}$$

$$f(x_i) = -\frac{8}{7}x_i + \frac{4}{63}x_i^3$$

$$C_{xx} = \frac{R}{R_{xx}}\frac{C_2}{C_1}, \quad C_{yx} = \frac{R}{R_{yx}}\frac{C_2}{C_1}$$

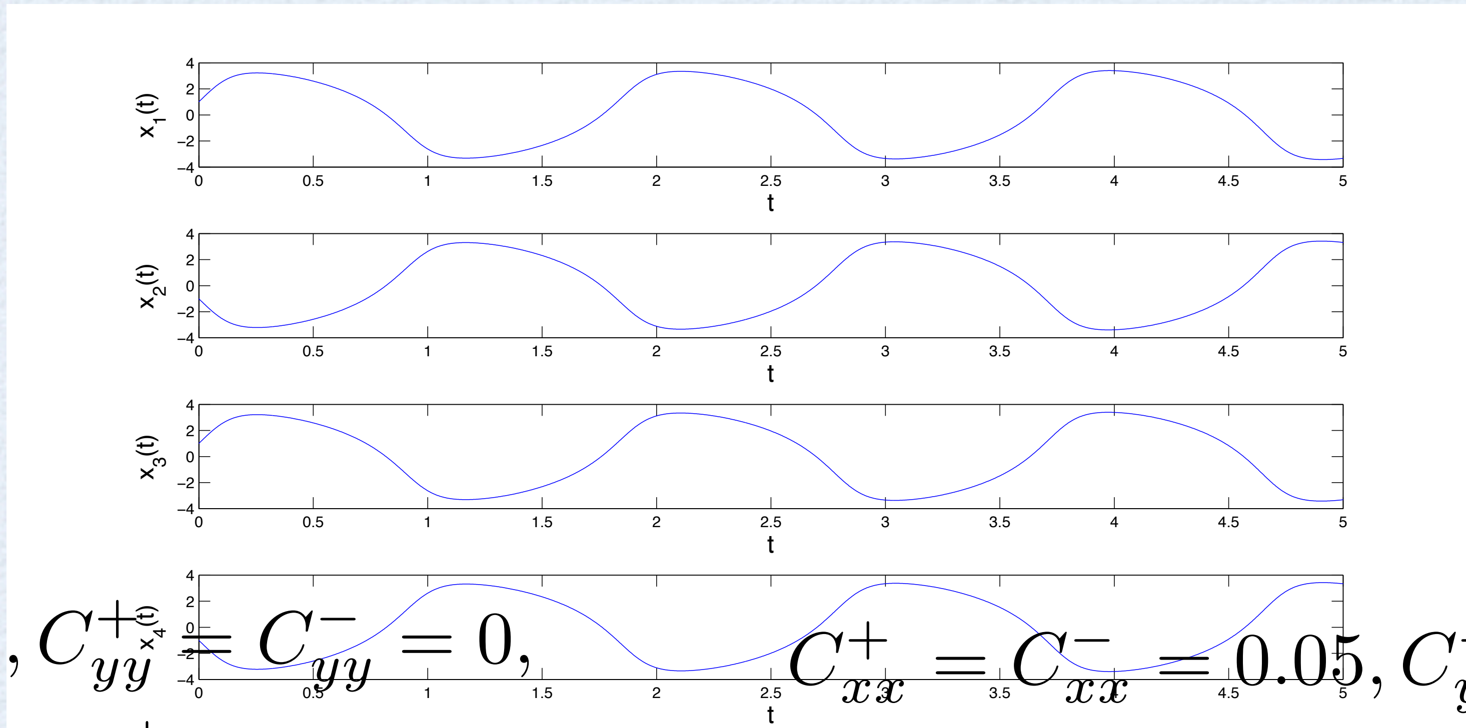
$$C_{xy} = \frac{R}{R_{xy}}, \quad C_{yy} = \frac{R}{R_{yy}}$$

$$\mathbf{R}^{-1}(p) = \begin{pmatrix} \frac{r^3-2C_{yy}^2 r}{\Delta} & \frac{r^2 C_{yy}-C_{yy}^3}{\Delta} & \frac{rC_{yy}^2}{\Delta} & \frac{C_{yy}^3}{\Delta} \\ . & \frac{r^3-C_{yy}^2 r}{\Delta} & \frac{r^2 C_{yy}}{\Delta} & \frac{rC_{yy}^2}{\Delta} \\ . & . & \frac{r^3-C_{yy}^2 r}{\Delta} & \frac{r^2 C_{yy}-C_{yy}^3}{\Delta} \\ . & . & . & \frac{r^3-2C_{yy}^2 r}{\Delta} \end{pmatrix}$$

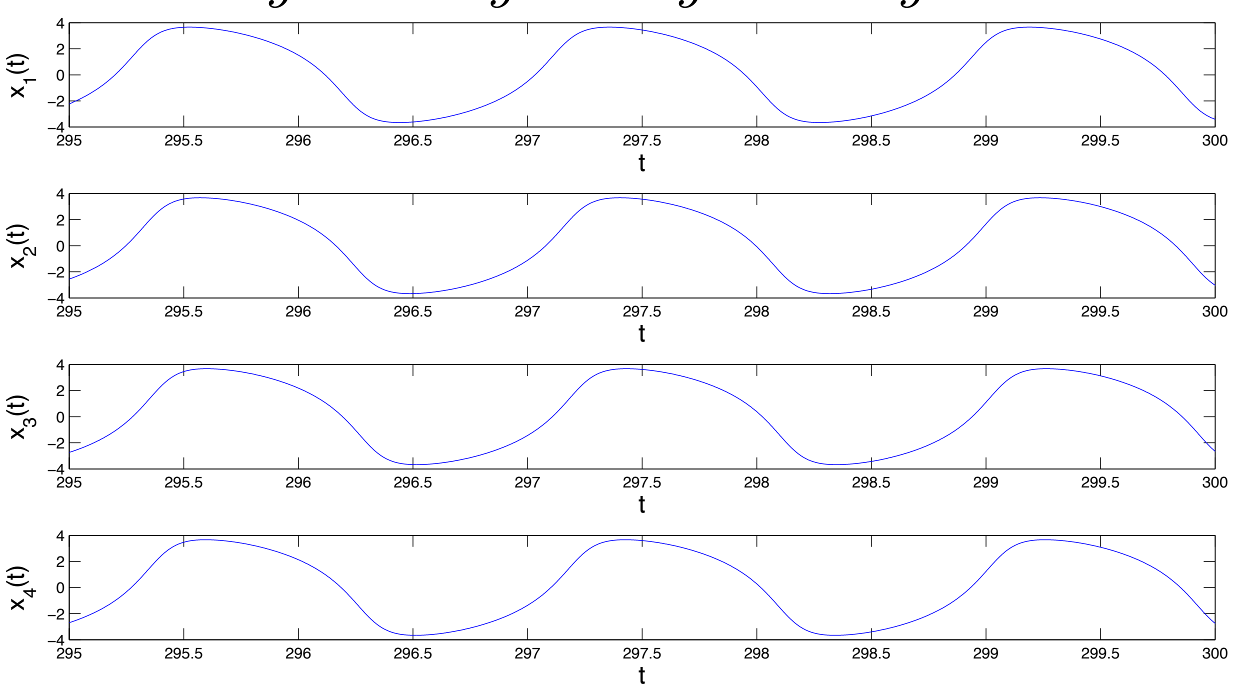
$$\Delta = r^4(p) - 3C_{yy}^2 r^2(p) + C_{yy}^4$$

$$r(p) = p + \sigma_y + 1 + \beta p^{-1}$$

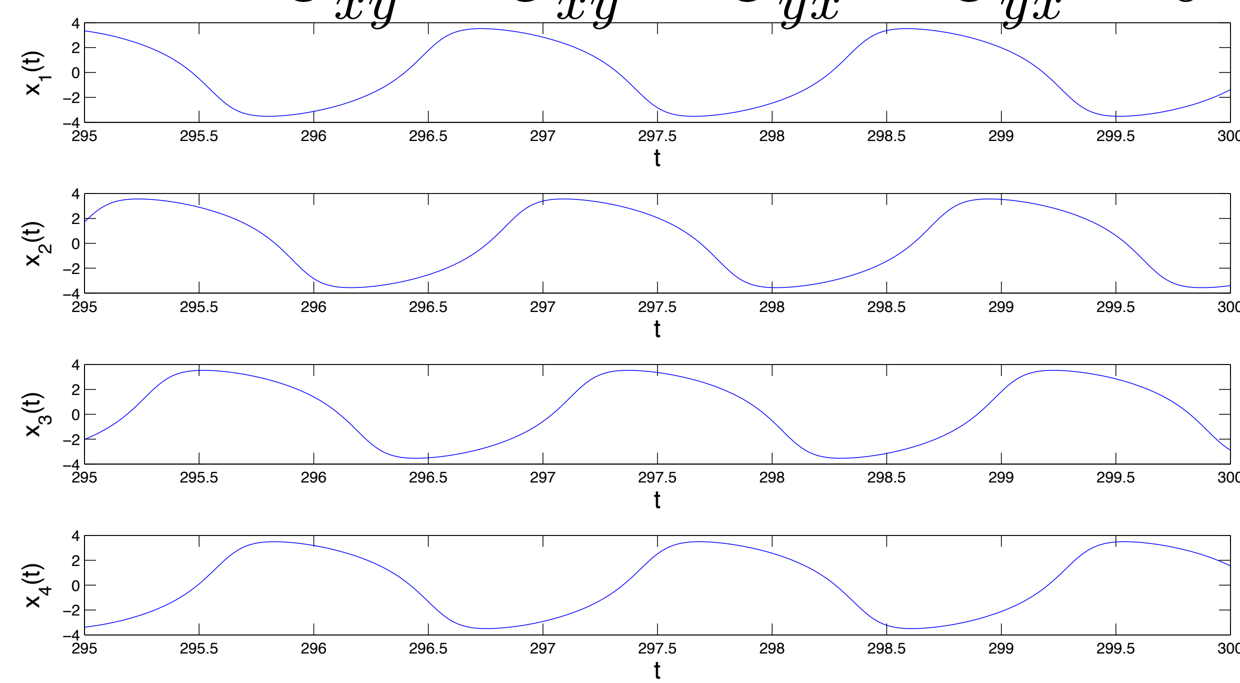
1-D array of 4 identical Chua's circuits



$$C^+_{xx} = C^-_{xx} = 0.05, C^+_{yy} = C^-_{yy} = 0,$$
$$C^+_{xy} = C^-_{xy} = C^+_{yx} = C^-_{yx} = 0$$



$$C^+_{xx} = C^-_{xx} = 0.05, C^+_{yy} = C^-_{yy} = 0.03,$$
$$C^+_{xy} = C^-_{xy} = C^+_{yx} = C^-_{yx} = 0$$



Conclusions and Perspectives

- The importance of the results provided in this work is twofold:
  - from the theoretical point of view it is shown that a FCON can be obtained by means of a suitable LCON, viz., there exists a procedure to derive a FCON from a particular LCON. In addition, these results provide some hints on how to investigate the inverse procedure.
  - from the practical point of view one may conceive simple prototype hardware platforms (based on local, space-invariant, memoryless and linear connections) whose computational properties are comparable to those of full connected networks.
- These results represent a solid starting point to design LCONs for processing spatial-temporal patterns without breaking them into frames.